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ADAPTED FOR THE USE OF

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ALGEBRA and TRIGONOMETRY.

LONDON: PRINTED BY
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AND PARLIAMENT STREET

THE

ELEMENTS

OF

ALGEBRA AND TRIGONOMETRY.

BY

WILLIAM N. GRIFFIN, B.D.

Sometime Fellow of S. John's College,
Cambridge.

LONDON:
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1871.

PREFACE.

THE PURPOSE of this book is to explain the rudiments of Algebra and Trigonometry to artisans and others, who may wish to be acquainted with them so far as to make the computations which arise in practice, and to read books in which science is treated mathematically. It is my hope that a student who masters this book and works its examples will find himself able to solve a large number of the questions which applied science raises, and to perform all the ordinary calculations which logarithms assist, and that he will find himself in possession of a trustworthy and available power, although, so far as I conduct him, he will have entered but a small portion of the wide field of modern analysis.

The Examples in this book are taken, with a few exceptions, from examination papers publicly accessible.

The reader who has no previous knowledge of the subjects of this work, is recommended to omit for a time the following articles :—

Algebra.—123–125. 255–260. 262–287.
Trigonometry.—Chap. IV. 117–119. 123–127. 130–132. 134 to the end of the chapter. 156 to the end.

W. N. GRIFFIN.

OSPRINGE VICARAGE:
January 4, 1871.

CONTENTS.

ALGEBRA.

LOGARITHMS.

TRIGONOMETRY.

ALGEBRA.

CHAPTER I.

THE MEANING OF ALGEBRAICAL SYMBOLS.

1. The method taken in this book is to conduct the reader into Algebra by regarding it as an extension of Arithmetic. The principles and processes of Arithmetic are therefore supposed to be already well known.

ALGEBRAIC SYMBOLS AND THEIR GENERALITY.

2. In Arithmetic, it will be remembered that symbols are used, digits as they are called, which have a certain generality, and mean different things as the unit is varied. The symbol 8, for instance, means eight units, but it may be eight shillings, eight pounds, eight yards, as may be intended. Operations are performed with these symbols, and true results are obtained, whatever be the unit in view. Eight and five added together make thirteen, whether they be pounds, or gallons, or inches which are thus being added together.

Now in Algebra representations of quantity are taken which have a further generality. Quantity is designated by a symbol, most commonly a letter of the alphabet, a, b,

c, *m*, *x*, for instance, and now not only the unit, but the number of units signified, waits to be assigned. The letter *a*, for instance, may represent any number referred to any unit. Operations will be performed with these letter representatives of quantity, and thereby problems will be solved for which the processes of arithmetic are insufficient.

3. *Obs.*—The word *number* in this book is to be taken to mean either an integer, or a quantity wholly or partly decimal.

4. Out of this statement of the purposes and method of algebra, many questions may at once arise in the reader's mind, how things so arbitrary as letters, so unconnected in themselves with magnitude, can have any definiteness, and can lead to any certain numerical results, how such symbols can be kept from confusion, how we are to know what each means. Such questions as these are noticed to assure the reader that his probable difficulties are foreseen and acknowledged, but they are questions which cannot be answered as yet. They will cease to give any difficulty as soon as algebra is seen in its application to practical questions. In the earlier pages of any treatise on algebra, the reader commencing the subject ought not to be surprised or discouraged if he sees but indistinctly the use of the processes in which he is being instructed. He will not be required, it is hoped, to accept any result without sufficient proof, but he may not see for a while the objects and purposes of the statements demonstrated.

5. Other symbols besides the italic letters of the alphabet are brought into use in algebra, to designate quantity. Affixes are sometimes attached to a letter, as a_1, a_2, a_3,...making thereby so many different symbols. The Greek alphabet, α, β, γ,...is also brought into service. A symbol of quantity in algebra is any mark which can be recognised and reproduced.

6. *Def.*—Algebraical symbols written down as a representation of magnitude, form what is termed 'an expression.'

THE POSITIVE AND NEGATIVE SIGNS.

7. Quantity, which can be expressed by any number, whole, fractional, or decimal, is designated in algebra by some letter of the alphabet or other easily written symbol. There are, furthermore, two signs which can be prefixed to the letter, one the positive, $+$, read 'plus,' another the negative, $-$, read 'minus,' and the power of these signs is that as one or other of them is prefixed to a representation of quantity, the subject whose magnitude is so represented has a contrary property or affection in some defined respect.

If $+a$ denotes a sum of money received, $-a$ can denote the same sum of money paid away, the quantity represented having these contrary properties or affections in respect of being given or received.

If $+a$ denotes a number of yards, as the distance which a person walks in one direction along an unlimited straight line, $-a$ can denote the same number of yards as the distance which he walks in the contrary direction.

If $+a$ denotes a number of years that are past, $-a$ may denote the same number of years yet to come.

If $+a$ means a number of feet which is the height of a point above the ground, $-a$ can mean the same number of feet as the depth of a point below the ground.

So, to extend this conception to apply to different algebraic symbols of quantity, if $+a$ denotes a sum of money received, $-b$ can denote a sum in magnitude b, paid away.

If $+a$ denotes an advance through a certain number of yards, $-b$ denotes a retreat through the number of yards expressed by b.

The positive and negative signs have thus an antagonistic and reversing power on the meaning of the symbols before which they stand. Of the two opposite characters which magnitude may have in some defined respect, it is immaterial which of the two signs is taken to represent one of those characters, the other sign representing the opposite

character. Thus, $+a$ or $-a$, may designate a height a, and then $-a$ or $+a$ will accordingly designate the same depth; $+a$ or $-a$ being taken to mean a sum of money a received, then $-b$ or $+b$ will respectively mean another sum of money b paid away.

8. *Obs.*—When a symbol stands without either algebraic sign prefixed to it, it is understood to bear the positive sign. Thus $+a+b+c$ is written $a+b+c$.

9. *Def.*—Quantities with the positive sign prefixed are called positive quantities. Quantities with the negative sign prefixed are called negative quantities.

10. *Def.*—By quantities of the same kind are meant such as can be referred to the same unit, quantities, for instance, of length, of weight, sums of money.

11. When the sign of a quantity is changed from positive to negative, or from negative to positive, the sign is said to be reversed.

12. The following signs are used as a kind of shorthand, to save writing:—

$\therefore$ means 'therefore.'
$\because$ „ 'since.'
$>$ „ 'greater than.'

Thus, $5>4$ means the statement that 5 is greater than 4.

$<$ means 'less than.'

Thus, $9<11$ means that 9 is less than 11.

ADDITION AND SUBTRACTION OF ALGEBRAICAL QUANTITIES.

13. In this view of the meaning of the two algebraical signs, the positive and the negative, it will be seen that quantities of the same kind (10) bearing the same sign, be it positive or negative, can be added one to another. The result of that addition is expressed by writing the algebraic symbols which express these quantities one after another in a horizontal line. Then if $+a$ and $+b$ mean separate

amounts of money received, $+a+b$ or $a+b$ (8) means the amount received altogether, the sum, as the word is understood in arithmetic. If $-c$ and $-d$ mean, in consistence with the former representations, separate amounts of money paid away, $-c-d$ means the amount paid away altogether. For example, if a be 32*l.*, b 26*l.*, c 34*l.*, d 20*l.*, when the positive sign is taken to designate money received, $a+b$ means 58*l.* received, and $-c-d$ means 54*l.* paid away.

If a, b, c, d continue to denote the same numbers, and these are now understood to mean numbers of miles, if $+a$ be the distance a man travels one day directly towards the east, $+b$ the distance he travels the next day in the same direction, $a+b$ will represent the 58 miles of his journeying towards the east, and consistently $-c-d$ can represent 54 miles of the journeying of some other man towards the west.

Again, quantities of the same kind bearing opposite signs can in virtue of the reversing power of these signs be subtracted one from another. The result of this subtraction is expressed by writing the symbols one after another with their proper algebraic signs in a horizontal line. Thus $+a$ denoting money which a person owes, and $-b$ consistently denoting money due to him, $+a-b$ or $a-b$ will mean what is called the balance of his account, or the difference between the debts due to him and the debts which he owes. If a is a greater quantity than b, $+a$ so absorbs $-b$ as to leave a positive excess, and $a-b$ means an excess of debt owing over money expected. On the contrary, if b exceeds a, $-b$ so absorbs $+a$ as to leave a negative excess, meaning that a balance remains of money expected.

14. *Obs.*—The order in which the symbols of quantity are written is immaterial. Thus $a+b-c$, $b+a-c$, $b-c+a$, $a-c+b$, $-c+a+b$, $-c+b+a$, all mean the same thing, namely, that after quantities a and b have been added together, the difference is taken between their sum and

another quantity c, and this difference is positive or negative, as $a+b$ is greater or less in magnitude than c.

In writing such an expression as this it is usual to place a positive quantity first, and the writing of its sign is dispensed with (8). Also when other reasons do not interfere, it is customary to write the algebraical symbols in the order in which they stand in the alphabet. Neither of these arrangements however is in any wise obligatory.

15. Hence $a-a$ or $-a+a$ has no magnitude, the one term meaning a magnitude operating with a reversing or destructive effect on another equal magnitude.

16. The sign $=$, read 'equals,' denoting equality of the quantities between which it stands, and the sign o, read 'zero' or 'naught,' denoting nullity or absence of all magnitude, are used in Algebra as in Arithmetic. Thus $a-a=0$. All quantities which admit of their magnitude being numerically expressed, being neither zero nor infinitely great, are called finite quantities.

17. Since magnitude is contemplated as in only two opposite states, two negative signs amount to the positive sign. For $-a$ expresses a quantity in a contrary condition to the same quantity $+a$. The reversing then of $-a$ throws it into the same condition with $+a$.

Thus,

$$--a = +a$$
$$---a = -a$$
$$----a = +a$$

and so on.

Questions for Exercise.

18. If a means 20 and b means 10 in the following questions,

1. What will be expressed when it is stated that a traveller has journeyed $a-b$ miles to the east? And what if he is said to have journeyed $-a+b$ miles towards the east?

2. If a man's expenses in one day are a£, what can $-b$£

consistently represent? What will be meant by stating that $b-a$ is his day's expenses?

3. If the boys added to a school after the holidays are a in number, what would be meant by $-a$ new boys? Would the school be made larger or smaller by receiving $-a+b$ new boys?

4. If a denote degrees in a thermometer above freezing point, what will $-b$ degrees mean?

5. If a boy wins a games over a playfellow, how may the latter be said to win $-a$ games? What would be meant by stating that one of them won $b-a$ games?

6. If a denotes the number of days a man works, would $-b$ denote a number of days when he was idle? Or what would be meant by stating that he worked $-b$ days?

Multiplication of Algebraic Quantities.

19. The expression $a+a$ means the addition of two quantities of the same magnitude and of the same sign. The result, therefore, is double of either of them, and is written $2a$. Similarly $a+a+a$ would be $3a$, and if b means the number of times that a is repeated,

$$a+a+a+\ldots \text{ would be } ba.$$

On this suggestion the result of multiplying together two quantities a and b, whether they mean whole numbers or fractions, is written ba or ab. Sometimes the sign of multiplication ($\times$) or a dot (.) is interposed between the letters, thus, $a\,.\,b$ or $a\times b$. The order of the letters, be it ab or ba, is immaterial, as we know in Arithmetic that when two quantities are multiplied together, it is immaterial which is regarded as the multiplier.

If ab is now viewed as a single quantity, the result of multiplying it by another quantity c will be written abc or cab, with or without the sign $\times$ or . above mentioned. This result being that of multiplying a, b, c together, and the order which the multiplication is performed being

immaterial, the product is indifferently written *abc*, *acb*, *bac*, *bca*, *cab*, *cba*.

20. *Def.*—In an expression such as *abc*, denoting the multiplication of two or more quantities together, any one of these is called the coefficient of the product of the others. Then *a* would be called the coefficient of *b* in *ab*, and of *bc* in *abc*, *c* the coefficient of *ab* in *abc*. Unity is the coefficient of *a* standing alone, though it is not usual to write 1*a*, but the letter *a* only.

21. *Def.*—Quantities multiplied together are called factors of the result. Thus *a*, *b*, *c*, are three factors of the quantity *abc*, which is produced by multiplying them together.

Law of Signs in Multiplication.

22. When two positive quantities, as *a* and *b*, are multiplied together, the product means the result of taking the multiple, or part of one designated by the other, and therefore is positive, since nothing in this process has altered the character or affection of either of the two quantities.

If a negative quantity, $-a$, is multiplied by a positive quantity, *b*, the multiple or part so taken of $-a$ is negative still, and thus the product is negative.

The negative sign having a power of reversing the character or affection of the magnitude to which it is prefixed, multiplying by a negative multiplier means the taking the multiple or part thus designated, and reversing besides the sign of the quantity multiplied. Hence:

1. If a positive quantity, *a*, is multiplied by a negative quantity, $-b$, the multiple or part expressed by *b* is taken, and the sign of *a* is reversed as well, whereby the result is $-ba$, or $-ab$.

2. If a negative quantity, $-a$, is multiplied by a negative quantity, $-b$, by the same reversal of sign the result is *ba* or *ab*.

These results are of the highest importance, and may be

thus placed in a tabular form. In the multiplication of two algebraic quantities

$$\begin{array}{cccccc} + & \text{multiplied by} & + & \text{gives} & +, \\ - & ,, & + & ,, & -, \\ + & ,, & - & ,, & -, \\ - & ,, & - & ,, & +, \end{array}$$

results which are expressed in words in the rule, that in multiplication 'like signs give $+$ and unlike signs give $-$.'

23. Conversely, the fact of the product of two quantities being positive certifies that these quantities are either both positive or both negative, and the product being negative certifies that the quantities have different algebraical signs.

Thus:
$$7 \times 5 \text{ or } (-7) \times (-5) = 35$$
$$7 \times (-5) \text{ or } (-7) \times (5) = -35.$$

The number 39 can arise as the product of 3 and 13, or of -3 and -13, while -39 can arise from the multiplication of 3 and -13 or of -3 and 13.

24. In the product ab, if either of the quantities a or b be zero, or have no magnitude, then whatever finite value the other may have, the product is zero, because if a, for instance, has no magnitude it cannot give a result of any magnitude by being taken any defined number of times.

Division of Algebraic Quantities.

25. If, in the ordinary arithmetical sense, a quantity a is divided by a quantity b, the result is expressed by writing these letters in the form of a fraction, $\frac{a}{b}$, the divisor being in place of denominator, and the dividend in place of numerator.

In division the law of signs above stated (22) holds good, the result $\frac{a}{b}$ being positive or negative as a and b have the same or different signs. If, for the sake of descrip-

tion, we use the terms dividend, divisor, and quotient, in the usual arithmetical sense, from the idea of division we know that the divisor and quotient multiplied together produce the dividend. Hence :

1. If the dividend is positive,
   the product of the divisor and quotient is positive,
   the divisor and quotient have the same sign (23);
   ∴ if the divisor is positive, so is the quotient,
   if the divisor is negative, so is the quotient;

and

2. If the dividend is negative,
   the product of the divisor and quotient is negative,
   the divisor and quotient have opposite signs (23);
   ∴ if the divisor is positive, the quotient is negative,
   if the divisor is negative, the quotient is positive.

To collect these results, when the divisor and dividend have the same sign, the quotient is positive; when the divisor and dividend have different signs, the quotient is negative.

Ex.					
	$\frac{7}{8}$	multiplied by	-3	gives	$-\frac{21}{8}$, or $-2\frac{5}{8}$
	$-\frac{3}{4}$	,, ,,	$\frac{1}{3}$	,,	$-\frac{1}{4}$
	$-\cdot 03$	,, ,,	-2	,,	$\cdot 06$
	$\frac{5}{3}$	divided by	-3	,,	$-\frac{5}{9}$
	$-\frac{5}{2}$	,, ,,	6	,,	$-\frac{5}{12}$
	$-\frac{5}{2}$	,, ,,	-6	,,	$\frac{5}{12}$.

26. Division is sometimes expressed in writing by the symbol $\div$, $a \div b$ meaning that a is divided by b; though $\frac{a}{b}$ is the more usual and the more convenient manner of expressing this operation.

Algebraical Fractions.

27. An expression such as $\frac{a}{b}$, wherein the result of a division is indicated without being performed, is an alge-

braical fraction. When a and b mean positive integers, it has the signification of a vulgar fraction in Arithmetic, meaning that when unity is divided into b equal parts, a of these parts are taken. In every case an algebraical fraction $\frac{a}{b}$ will mean a quantity which, if it be multiplied by b, will give the result a.

In the fraction $\frac{a}{b}$, the quantity a is called, as in Arithmetic, the numerator, and b the denominator.

28. If a and b have the same signs the fraction is positive (25), if different signs it is negative. If the sign of either a or b is changed, the sign of the fraction is changed, inasmuch as if they previously had the same, they now have different signs, and the fraction is changed from being positive to be negative. If a and b had previously different signs, they have now the same, and the fraction, from being negative, becomes positive.

29. If the numerator of a fraction be zero while the denominator is finite, the fraction has no magnitude or is zero. For if it had any magnitude, the multiplication of that magnitude by the denominator must produce a result different from zero the numerator.

If the numerator be finite and the denominator zero, the fraction is then beyond numerical representation. For if it were supposed to have any defined magnitude, this magnitude, multiplying the denominator, could still produce no result but zero, and could not therefore give the numerator.

Involution.

30. *Def.*—The result of multiplying an algebraic quantity by itself is called its square or second power. Thus, $a \times a$ is written a^2, and is called the square or second power of a, a itself being, by analogy, called the first power of a.

So the square multiplied by a, or the product $a \times a \times a$, is called the cube or third power of a, and is written a^3.

These terms have been suggested by the geometrical facts that if a be the number of units of length in a line, suppose 4 inches, then 4×4, or 16, is the number of square inches in the square described on that line, and $4 \times 4 \times 4$, or 64, is the number of cubic inches in the cube of which this line is an edge.

By extending this notation, $a \times a \times a \times a$ is written a^4, and called the fourth power of a, and generally if there be any number of times m, that the quantity a is repeated in continued multiplication, the result is called the mth power of a, and is written a^m.

These operations are sometimes termed the raising a to the second, third, fourth, &c., powers, and the number expressing that power is called the index or exponent. Thus, 5 is the index or exponent of a in a^5, which is called a raised to the 5th power.

31. When the quantity raised to a power is negative, its square is positive by the law of signs (22), its cube is negative, its fourth power is positive, and generally the power is positive or negative as the exponent is even or odd respectively.

EVOLUTION.

32. *Def.*—A quantity which multiplied by itself gives the result a, is called the square root of a, and is written $\sqrt{a}$ or $a^{\frac{1}{2}}$. Thus, $\sqrt{a} \times \sqrt{a} = a$ or $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a$.

Obs.—The symbol $\surd$ is supposed to be a perverted form of the letter r, the initial letter of the word *root.*

A quantity whose cube or third power is a is called the cube root of a, and is written $\sqrt[3]{a}$, or $a^{\frac{1}{3}}$.

By extension of this notation $\sqrt[m]{a}$, or $a^{\frac{1}{m}}$, means a quantity which, raised to the mth power, produces a.

33. A negative quantity can have no numerical square root, because any numerical quantity, be it positive or

negative, can never produce, when multiplied by itself, any but a positive result. So, neither can a negative quantity have a numerical fourth, sixth, or any even root.

A positive quantity has two numerical square roots of different signs ; for since either $a \times a$ or $-a \times -a$ produces a^2, a or $-a$ is thus the square root of a^2. So also will a positive quantity have two numerical fourth, sixth roots, and so on, equal in magnitude but opposite in sign.

Thus, either 8 or -8 is the square root of 64, 2 or -2 is the fourth root of 16, 2 or -2 is the sixth root of 64.

A quantity has but on enumerical cube root of the same sign as its own, so also only one numerical fifth or seventh root of the same sign as its own. Thus, 2 is the cube root of 8, the fifth root of 32.

34. *Def.*—An expression (6) not connected with any other by the signs + or —, as a, $5ab$, xyz^2, is said to be 'of one term,' and is called a monomial. But where two quantities are united by the signs + or —, as a^2+36, $xy-4z^2$, $1+x^3$, these quantities are called the 'terms' of the expression, and the expression is said to consist of two terms, or to be a binomial. Expressions wherein more than two terms are connected by the signs + or — are called multinomials or polynomials.

35. *Brackets.*—For the purpose of contemplating a binomial or polynomial as a single quantity, and subjecting it to operation as a single quantity, its terms are enclosed in 'brackets' and written thus, $\{a-2b\}$, $[a^2-3ab+b^2]$. A bar over the expression, thus, $\overline{a-2b}$, has the same meaning, which is this, that the quantity b being doubled, and having its sign reversed, is combined, in way of addition, with the quantity a, and the final result, whatever it be, is contemplated under the symbol $[a-2b]$.

36. By a bracket, the sign of involution is made to represent the result of multiplying an expression any assigned number of times by itself. Thus, $(a+b)^2$ means the result of multiplying the quantity $a+b$, regarded as a single

expression, by itself, whereas $a+b^2$ would mean that b only is to be multiplied by itself, or squared, and the square thus obtained is to be added to a.

37. The sign of evolution (32) is extended by a horizontal line, or bar, to affect all the quantities under the bar. Thus, $\sqrt{a+b-c}$ means the same as $\sqrt{(a+b-c)}$, namely, that when a and b have been united by addition, and c taken from the sum, the square root of the result is taken.

38. The bar which forms a fraction has the effect of a bracket, signifying that all the terms above it combine to make the numerator, while all the terms below combine to form the denominator (27). Thus, $\frac{a^2+b^2}{a+b}$ means that a and b are separately squared (30), and that the squares added together make the numerator, while a and b added together make the denominator, and the value of the fraction is the quantity which, multiplied by $(a+b)$, gives for the result (a^2+b^2).

39. Hence, brackets serve to express the result of operating on the results of previous operations. Thus, $c(a^2+b^2)$ means that after the result a^2+b^2 has been formed, that result is multiplied by c. The expression is thus equivalent to ca^2+cb^2, inasmuch as the result must be the same whether a and b are separately squared, and the sum of their squares multiplied by c, or these squares are separately multiplied by c and the two products added together.

As another instance, $(a-2b+c^2)^2$ means that c is squared, and the square added to a, then the double of b subtracted, and the result finally squared.

40. Thus, $a+b\,.\,c+d$ means that b is to be multiplied by c, and the product combined, by addition, with a and d.

$(a+b)c+d$ means that after a and b have been added together, their sum is multiplied by c, and d is added to the result of that multiplication.

$(a+b)(c+d)$ means that after a and b are added together

as one quantity, and c and d are added together as another quantity, these two quantities are multiplied together, and the form expresses the result of that multiplication.

$$\text{If } a=2,\ b=3,\ c=3,\ d=5,$$
$$a+bc+d=2+9+5=16,$$
$$(a+b)c+d=15+5=20,$$
$$(a+b)(c+d)=40.$$

41. *Caution.*—Observe the difference between ab^2, a^2b, and $(ab)^2$.

ab^2 means that b is squared and the result multiplied by a.

a^2b means that a is squared and the result multiplied by b.

$(ab)^2$ means that after a and b have been multiplied together, the result is squared. Hence, $(ab)^2$ being $ab\times ab$, or $a\times a\times b\times b$, is the same as a^2b^2.

$$\text{If } a=2,\ b=3,$$
$$ab^2=18,\ a^2b=12,\ (ab)^2=36.$$

42. A negative sign before a bracketed expression reverses the sign of every term within the bracket:

thus, $-(a-bc+d^2)$ means $-a+bc-d^2$.

A negative sign before a fraction has a similar effect on every term of the numerator:

thus, $-\dfrac{a-bc+d^2}{a^2-b^2}=\dfrac{-a+bc-d^2}{a^2-b^2}$ or $\dfrac{bc-a-d^2}{a^2-b^2}$.

Obs.—A negative sign before a fraction alters the sign of every term of *either* the numerator or the denominator, but not of both. Alteration of the signs of both is no alteration at all, because $\dfrac{a}{b}$ is the same quantity as $\dfrac{-a}{-b}$, the quantity which multiplied by b gives a, being the same as that which multiplied by $-b$ gives $-a$ (22). Hence, the negative sign, meaning to reverse the sign of the fraction $\dfrac{a}{b}$, will accomplish this by reversing the sign of *either* a or b.

To give a numerical instance, $\frac{14}{2}$ is 7, but $\frac{-14}{2}$ or $\frac{14}{-2}$ is -7, inasmuch as 2 and -7 multiplied together make -14 (22), and -2 and -7 multiplied together make 14. But $\frac{-14}{-2}$ is 7, because 7 and -2 multiplied together give -14. Thus, reversal of the signs of *both* numerator and denominator makes no alteration in a fraction, but reversal of the sign of *either* one of them reverses the sign of the fraction.

As an algebraical example :

$$\begin{aligned} \frac{a-b+c}{a+b-c} &= -\frac{b-a-c}{a+b-c} \\ &= -\frac{a-b+c}{c-a-b} \\ &= \frac{b-a-c}{c-a-b}. \end{aligned}$$

It is a very useful exercise to verify these algebraical statements, and others which may arise, by giving to the letters any numerical values taken at random ; for instance, $a=7$, $b=4$, $c=2$.

43. To describe in words the operations indicated by the quantity

$$\sqrt{\frac{(a^3-b)c^2}{(a+b)^3}}$$

1. The quantity a is cubed (30).

2. The quantity b has its sign reversed, and is after this reversal added to a^3.

3. The quantity c is squared, and the square obtained is used as a multiplier of the result of 2.

4. a is added to b, and the sum is cubed (36).

5. The result of 3 being taken as a numerator, and that of 4 as a denominator, a fraction is formed.

6. The square root of this fraction is taken.

If a is 3, b is 12, and c is 5, the expression has for its numerical value $\frac{1}{3}$.

44. This chapter shall be closed by reviewing, in the form of a table, the symbols used in Algebra to abridge writing or describe operations.

$=$ means that the quantities between which it stands are equal to one another.

$>$ means 'greater than,' and $<$ means 'less than,' the opening being towards the quantity to be designated as the greater of the two which are compared.

$\therefore$ is a symbol to stand for the word 'therefore.'

$\because$ is a symbol to stand for the word 'since.'

$+$ and $-$ are the antagonistic signs, meaning that the quantities before which they stand have contrary qualities in some particular respect.

$\times$ or . is the sign to mean multiplication of quantities between which it stands.

$\div$ means division, the former of the quantities between which it stands divided by the latter.

$\sqrt{}$ means the square root of the quantity to which it is prefixed, $\sqrt[3]{}$ the cube root.

45. Examples of the power of brackets :

1. $a-(b-c)+c-d=a-b+c+c-d.$
2. $a+b-(a^2-b^2+c^2)-(ab+bc-ac)$
$$=a+b-a^2+b^2-c^2-ab-bc+ac.$$
3. $a-\{3a-5b-(4a-3b+c)-d\}$
$$=a-3a+5b+(4a-3b+c)+d$$
$$=a-3a+5b+4a-3b+c+d.$$

If $a=4$, $b=3$, $c=-2$, the following statements may be verified, these numerical values being substituted for the letters.

4. $3ab+5ac-a^2+c^2=36-40-16+4=-16.$
5. $(a^2-b^2+c^2)(a+b-c)-(a^2-b^2-c^2)(a-b+c)$
$$=11\times 9+3\times 1=102.$$
6. $(a+b+c)(3a-4b)=0\ (24).$

7. $(a-b)^3-(b-c)^2-(a-c)=1-25-6=-30.$

8. $a^2-(a-c)+\dfrac{a^2-b^2}{a-b}=16-6+\dfrac{16-9}{4-3}=10+\frac{7}{1}=17.$

9. $5a^2b+5ac^2-\left(\dfrac{a-b}{a+b}-\dfrac{b-c}{b+c}\right)=240+80-\frac{1}{7}+\frac{5}{1}=324\frac{6}{7}.$

10. $a-\sqrt{5(b-c)}=4-\sqrt{5\times 5}=4-5=-1.$

11. $\sqrt{a^2+b^2}-6c=\sqrt{16+9}+12=5+12=17.$

12. Find the value of $x^2-y^2-(x-y)^2+a\{x-(y-z)\}$, when $x=4$, $y=3$, $z=1$, $a=2$.

13. Describe in words the operations indicated in the expressions (43):

$$(a^2+b)(a+b^2).$$

$$a^3b-ab^3.$$

$$\sqrt{3(a+b)(a^2+b^2)}.$$

$$\sqrt[3]{\frac{5(a^3+b)}{4(a^2-b)}}.$$

14. If $a=1$, $b=2$, $c=3$, $d=4$, prove that the two expressions

$$\{d-(c-b+a)\}\ \{d+c-(a+b)\}$$

and
$$d^2-(c^2+b^2)+a^2+2(bc-ad)$$

have the same value.

CHAPTER II.

OPERATIONS WITH ALGEBRAICAL SYMBOLS.

46. The meaning of algebraical symbols having been explained in the preceding chapter, the methods of performing algebraical operations have now to be described. These operations are reducible to the three divisions:

1. Addition and subtraction.
2. Multiplication and division.
3. Involution and evolution.

ADDITION AND SUBTRACTION.

47. When the same letter, expressing quantity, appears with numerical coefficients (20), as $5a$, $3a$, $7a$, and thus presents two or more terms to be added together, the addition of these is effected by adding the numerical coefficients, and making their sum the coefficient of the literal symbol.

Thus:
$$5a+3a=8a,$$
$$5a+3a+7a=15a.$$

To see the reason of this, suppose that a denotes some defined weight, then $5a$ means five such weights, and $3a$ means three of them, and these together make eight of these weights, or $8a$. So also five, three, and seven such weights make together fifteen of them, or $15a$.

48. But when different letters expressing quantity appear with any coefficients, as a, $3b$, $5c$, their addition can only be left expressed by writing them in a line, $a+3b+5c$. For according to the former illustration, if a denote a certain weight, b, c certain other weights, it is impossible to exhibit the sum in any simpler form, unless we know what are the several weights designated, and these are not defined as long as a, b, c remain general.

49. Similarly, when the same letter stands with numerical coefficients, and one quantity thus designated is to be subtracted from the other, if this latter be accounted positive, the former is negative with respect to it, and the result comes from taking the difference of the coefficients, and attaching that difference as the coefficient of the letter. Thus, $5a-2a=3a$, $4b-6b=-2b$. If a, for instance, denote a space travelled, $5a$ means five such spaces travelled in one direction, say to the east, $-2a$ then means two such spaces travelled towards the west, and $3a$ is the three such spaces whereby, on the whole, the traveller advances to the east. So if b be some other space, and the westward motion $6b$ exceeds the eastward motion $4b$, the result $-2b$ shows that the traveller advances though two spaces b towards the west.

50. Different letters signifying quantity can only be represented as one subtracted from the other, or one antagonistic to the other, by writing them in a line with contrary signs, as $a-b$, $3a-4b$.

51. When any one of the quantities to be added or subtracted is a binomial or multinomial (34), it must be released from its bracket, and the preceding remarks show how it can be combined with other quantities in addition or subtraction, either expressed or effected.

Ex. 1. Thus,
$$\begin{aligned} & a-b+3a+c-(2a+5c-b) \\ &= a-b+3a+c-2a-5c+b \ (42) \\ &= 2a-4c. \end{aligned}$$

Ex. 2.
$$\begin{aligned} & a-[2a-\{3b-(a-c)\}+(2a-b)] \\ &= a-[2a-3b+(a-c)+2a-b] \\ &= a-[2a-3b+a-c+2a-b] \\ &= a-2a+3b-a+c-2a+b \\ &= -4a+4b+c. \end{aligned}$$

Ex. 3.
$$\begin{aligned} & 3a-b-[2a-(3a-5b)+\{5b-(c-2a)\}] \\ &= 3a-b-[2a-3a+5b+5b-(c-2a)] \\ &= 3a-b-[2a-3a+5b+5b-c+2a] \\ &= 3a-b-2a+3a-5b-5b+c-2a \\ &= 2a-11b+c. \end{aligned}$$

Ex. 4. Let it be required to add together the three quantities :

$$\frac{x^2}{4}-\frac{y^2}{6}+\frac{z^2}{6},\ \frac{y^2}{4}-\frac{z^2}{6}+\frac{x^2}{8},\ \frac{z^2}{6}+\frac{x^2}{6}+\frac{y^2}{8},$$

and to subtract from the result, $z^2-x^2+\frac{y^2}{2}$.

When the first three quantities are added together,

the coefficient of x^2 is $\frac{1}{4}+\frac{1}{8}+\frac{1}{6}=\frac{13}{24}$,
" " y^2 is $-\frac{1}{6}+\frac{1}{4}+\frac{1}{8}=\frac{5}{24}$,
" " z^2 is $\frac{1}{6}-\frac{1}{6}+\frac{1}{6}=\frac{1}{6}$,

and the result is accordingly

$$\tfrac{13}{24}x^2+\tfrac{5}{24}y^2+\tfrac{1}{6}z^2.$$

If now $z^2-x^2+\frac{y^2}{2}$ be subtracted from this result, or $-z^2+x^2-\frac{y^2}{2}$ added to it, we have finally,

$$\tfrac{13}{24}x^2+x^2+\tfrac{5}{24}y^2-\tfrac{1}{2}y^2+\tfrac{1}{6}z^2-z^2,$$

$$\text{or } \tfrac{37}{24}x^2-\tfrac{7}{24}y^2-\tfrac{5}{6}z^2.$$

52. *Examples for Practice in Addition and Subtraction.*

1. Add together the four quantities :

$$7x+3y+8z-4,\quad 3y-5x+6-2x,$$
$$5z-7+3x-8y,\quad 2-4x+3y-2z.$$

The sum is $-x+y+11z-3$.

2. Add together the four quantities :

$$2x^3+5ax^2-7a^2x+4a^3,\quad -3x^3+2ax^2-4a^2x-6a^3,$$
$$x^3-3ax^2+5a^2x+2a^3,\quad -5x^3-4a^2x+2ax^2-3a^3.$$

The sum is $-5x^3+6ax^2-10a^2x-3a^3$.

3. From $2x^2-5y^2+7z+a$ take $3z+4x^2-2y^2+b$.

The remainder $=4z-2x^2-3y^2+a-b$.

4. From $8ax^2y+5by^2z-7az^4-5b^5+6c^5$
take $3a^2xy-4b^2yz-3az^4+2b^5+3c^5$.

The remainder $=8ax^2y-3a^2xy+5by^2z+4b^2yz$
$-4az^4-7b^5+3c^5$.

5. Add together the four quantities :

$$2y^3-4y^2z+2yz^2-3z^3,\ 7z^3+2yz^2-3y^2z-4y^3,$$
$$-5y^3+y^2z-8yz^2-z^3,\ -3z^3-4yz^2+2y^2z+7y^3.$$

The sum is $-4y^2z-8yz^2$.

6. Reduce to the simplest form the expression :

$$3x-5y-(4x-5y)-(4x+5y)-\{7x+3y-(5x-2y)\}.$$

Ans. $-7x-10y$.

7. Reduce to the simplest form the expression :

$$3a^2-2ab+b^2-(-4a^2+7ab+3b^2)+(b^2-2bc+7c^2)$$
$$-(a^2+b^2+c^2).$$

Ans. $6a^2-9ab-2b^2-2bc+6c^2$.

8. $3x-2a-(2x+4a-9+2b)+12b-5a+(8a-\overline{2b-c})$
$-\{a-(4b-\overline{x+2c})\}=-4a+12b-c+9.$

9. $p+q-2r-(2p-3q-4r)=-p+4q+2r.$

10. Simplify $3(a-b+c)-5(a-2b+3c)+4(a-3b+2c)$
$-2(a-7b-2c)$.

Ans. $9b$.

11. $a^3+3a^2x+3ax^2+x^3-(a^3-3a^2x+3ax^2-x^3)$
$=6a^2x+2x^3.$

12. $xy-(x^2y-xy^2)-(xy^2-3xy)=4xy-x^2y.$

13. $x^4+6x^2y^2-(4x^3y-4xy^3)+y^4-3x^2y^2-4xy^3$
$=x^4-4x^3y+3x^2y^2+y^4.$

14. $m^2-2mn+n^2-(m^2+2mn+n^2)=-4mn.$

15. Reduce to the simplest form :

$$a^2+2d^2-(2e^2-b^2)-\{(d^2-e^2-c^2)+(d^2-e^2)\}.$$

Ans. $a^2+b^2+c^2$.

53. It is instructive for the reader to verify examples for himself in the particular cases when the symbols bear certain numerical values, which he may assign to them at pleasure. For instance, in example 14, let $m=3$, $n=2$,

then $m^2-2mn+n^2-(m^2+2mn+n^2)$
$$=9-12+4-(9+12+4)=-24;$$
and $-4mn=-24$;
or, as another case, let $m=\frac{1}{5}$, $n=-\frac{1}{2}$,
then, $m^2-2mn+n^2-(m^2+2mn+n^2)$
$$=\tfrac{1}{25}+\tfrac{1}{5}+\tfrac{1}{4}-(\tfrac{1}{25}-\tfrac{1}{5}+\tfrac{1}{4})=\tfrac{2}{5},$$
and $-4mn=\frac{2}{5}$.

Multiplication.

54. It has been explained that the multiplication of algebraical quantities is expressed by writing them together as if they made a word (19), sometimes, though not of necessity, with the symbol × or . between them. It has also been seen that the order in which the quantities multiplied together are thus continuously written is immaterial (19). In certain cases, as shall now be shown, the multiplication thus expressed can in a measure be effected, and a product obtained simpler in form.

55. When different literal symbols carrying numerical coefficients (20) are to be multiplied together, the product is expressed by writing the literal symbols together, and making the product of the numerical symbols the coefficient of the result. Thus, $4a$ multiplied by $3b$ gives $12ab$. For since a multiplied by b gives the product ab, $4a$ multiplied by b gives the four terms

$$ab,\ ab,\ ab,\ ab, \text{ or } 4ab,$$

and $4a$ multiplied by $3b$ gives this latter product taken three times, $4ab$, $4ab$, $4ab$, or makes $12ab$.

In this, and all examples of multiplication, the law of signs (22) has to be constantly remembered. Thus, $3a$ multiplied by $-5b$ gives $-15ab$.

56. When a literal symbol, raised to any power, has to be multiplied by the same symbol, raised to any power, the

product is obtained by adding their indices (30). This is a statement generally true, but at present, with a view to the purposes of this book, it will not be demonstrated further than with indices which are positive integers.

It has been seen that a^2 means $a\times a$, a^3 means $a\times a\times a$, and so on, and if m be any integral number,

$$a^m \text{ means } a\times a\times a\times \ldots \text{ taken } m \text{ times.}$$

Hence $a^m\times a$ means $a\times a\times \ldots \times a$ taken $m+1$ times, or a^{m+1}.

So $a^m\times a^2$ means a^{m+2},

and generally $a^m\times a^n$ is a^{m+n}.

Thus two powers of the same letter are multiplied together by adding their indices.

So $a^m\times a^n\times a^r=a^{m+n+r}$.

57. By combination of the results of the two last articles, it will be seen how different powers are multiplied together when they also have numerical coefficients.

Thus,
$$5a^2\times 3a^3=15a^5.$$
$$7a^p\times 5a^4=35a^{p+4}.$$
$$-3a^3\times 4a^3=-12a^6.$$
$$(-3a^3)\times(-6a^5)=18a^8.$$

Hence also
$$5a^2b\times 3abc=15a^3b^2c.$$
$$-ac\times 4a^2b=-4a^3bc.$$
$$3x^m\times(-4x^2y)=-12x^{m+2}y.$$

58. When a monomial multiplies a quantity of more than one term (34), its action on each term of the latter quantity is taken, and the results collected. It has been seen (39) that a product expressed as $a(b+c)$, means that a is separately multiplied with b and c, and the results added. Hence $a(5a+a^2)$, meaning that a is to multiply $5a$ and a^2 separately, gives the successive terms $5a^2$ (57) and a^3 (56), and the final result $5a^2+a^3$.

So $$3a^2(4ab-5c)=12a^3b-15a^2c.$$
$$3a^4(4ab-5b^2-c)=12a^5b-15a^4b^2-3a^4c.$$
$$5x(x^2-3xy+y^2)=5x^3-15x^2y+5xy^2.$$
$$3a^2(x-\sqrt{xy}+y)=3a^2x-3a^2\sqrt{xy}+3a^2y.$$
$$(x^2-7xy+4y^2)\times(-3x)=-3x^3+21x^2y-12xy^2.$$

59. When expressions each of more than one term are multiplied together, consideration of the meaning of the operation expressed will lead to the process of effecting it. Thus, $(a+b)(c+d)$ means that $c+d$ is to be multiplied by a, and also by b, and the results added together.

Now $$a(c+d)=ac+ad,$$
$$b(c+d)=bc+bd,$$
$$\therefore\ (a+b)(c+d)=ac+cd+bc+bd.$$

The same result would have arisen, since the order in which the terms stand is immaterial (19), if $a+b$ were regarded as multiplied by $c+d$.

Similarly, $(a-b)(c+d)$ would mean that after $c+d$ has been multiplied by a, and then by b, the difference of the results is to be taken.

$$\therefore\ (a-b)(c+d)=ac+ad-bc-bd.$$
So $$(a-b)(c-d)=ac-ad-bc+bd.$$

It will be observed how the law of signs (22) operates to give a positive sign, in the result, to that term which arises from the multiplication of terms with like signs in the two quantities multiplied together, and a negative sign when those signs are unlike.

60. By the law of signs, when an expression consists of two factors (21), if the sign of both be reversed the product is unaltered, if the sign of one only be reversed the sign of the product is reversed.

Thus, $$(-a)\times(-b)=ab,$$
but $$(-a)\times b \text{ or } a\times(-b)=-ab.$$
Thus, $$(a-b)(c-d)=(b-a)(d-c),$$
but $$(a-b)(c-d)=-(a-b)(d-c)$$
$$(a-b)(c-d)=-(b-a)(c-d).$$

61. Ex. 1. The expression $(a^2-a)(5a^3-2a^2)$ means that $5a^3-2a^2$ is first to be multiplied by a^2, and then by a, and the result of the latter operation subtracted from, or made antagonistic to, the result of the former.

Now $$a^2(5a^3-2a^2)=5a^5-2a^4 \quad (58)$$
$$a(5a^3-2a^2)=5a^4-2a^3.$$

Hence the final result is

$$5a^5-2a^4-5a^4+2a^3=5a^5-7a^4+2a^3 \quad (49).$$

It would have been equally just to regard the expression $(a^2-a)(5a^3-2a^2)$, as meaning that a^2-a was to be multiplied by $5a^3$, then by $2a^2$, and the result of the latter operation subtracted from that of the former. The first result being

$$5a^3(a^2-a)=5a^5-5a^4 \quad (58),$$

and the second being

$$2a^2(a^2-a)=2a^4-2a^3,$$

the final result is as before

$$5a^5-5a^4-2a^4+2a^3=5a^5-7a^4+2a^3 \quad (49).$$

Ex. 2. Let $5x^3-7x^2y-3xy^2+y^3$ be multiplied by $x-\frac{1}{3}y$.

$$\begin{aligned}&(5x^3-7x^2y-3xy^2+y^3)(x-\tfrac{1}{3}y)\\ &=(5x^3-7x^2y-3xy^2+y^3)x\\ &\quad-(5x^3-7x^2y-3xy^2+y^3)\frac{y}{3}\\ &=5x^4-7x^3y-3x^2y^2+xy^3\\ &\quad-\tfrac{5}{3}x^3y+\tfrac{7}{3}x^2y^2+xy^3-\frac{y^4}{3}\\ &=5x^4-\tfrac{26}{3}x^3y-\tfrac{2}{3}x^2y^2+2xy^3-\frac{y^4}{3}.\end{aligned}$$

62. Ex. 3. To multiply $\frac{3}{2}x^3-5x^2+\frac{x}{4}+9$ by $\frac{x^2}{2}-x+3$.

$$\begin{aligned}&\left(\tfrac{3}{2}x^3-5x^2+\frac{x}{4}+9\right)\left(\frac{x^2}{2}-x+3\right)\\ &=\frac{x^2}{2}\left(\tfrac{3}{2}x^3-5x^2+\frac{x}{4}+9\right)-x\left(\tfrac{3}{2}x^3-5x^2+\frac{x}{4}+9\right)\\ &\quad+3\left(\tfrac{3}{2}x^3-5x^2+\frac{x}{4}+9\right),\end{aligned}$$

$$\begin{aligned}
&=\tfrac{3}{4}x^5-\tfrac{5}{2}x^4+\tfrac{1}{8}x^3\ \ +\tfrac{9}{2}x^2\\
&\qquad\quad -\tfrac{3}{2}x^4+5x^3\ \ -\tfrac{1}{4}x^2\ \ -9x\\
&\qquad\qquad\qquad +\tfrac{9}{2}x^3\ \ -15x^2+\tfrac{3}{4}x\ \ +27\\
&=\tfrac{3}{4}x^5-4x^4+7\tfrac{7}{8}x^3-4\tfrac{3}{4}x^2-3\tfrac{3}{4}x+27.
\end{aligned}$$

63. When two expressions of any length have to be multiplied together, it sometimes leads to clearness to arrange them in the manner of a multiplication sum in numbers, with this difference, that in Algebra we work from left to right instead of from right to left. Suppose, for instance, that $a^2-2ax-x^2$ and $a^2+2ax-x^2$ have to be multiplied together. The process may be placed in this form :—

$$\begin{array}{rl}
a^2(a^2-2ax-x^2) & =a^4-2a^3x-a^2x^2\\
+2ax(a^2-2ax-x^2) & \qquad +2a^3x-4a^2x^2-2ax^3\\
-x^2(a^2-2ax-x^2) & \qquad\qquad\qquad -a^2x^2+2ax^3+x^4\\
 & =a^4-6a^2x^2+x^4,
\end{array}$$

the terms which contain a^3x and ax^3 destroying one another; or in this form:—

$$\begin{array}{lllll}
a^2 & -2ax & -x^2 & & \\
a^2 & +2ax & -x^2 & & \\
\hline
a^4 & -2a^3x & -a^2x^2 & & \\
 & +2a^3x & -4a^2x^2 & -2ax^3 & \\
 & & -a^2x^2 & +2ax^3 & +x^4\\
\hline
a^4 & & -6a^2x^2 & & +x^4.
\end{array}$$

It is in very long operations that there is some convenience in thus arranging in vertical rows the terms which can be combined. The student, however, should gain as early as possible the power of readily writing out the result of multiplication in a horizontal line.

64. The following results are of frequent occurrence in operation, and are examples of multiplication :—

$$(a+b)(a-b)=a^2+ab-ab-b^2=a^2-b^2.$$

$$\begin{aligned}(a^2+ab+b^2)(a-b)&=a^3+a^2b+ab^2-a^2b-ab^2-b^3\\&=a^3-b^3.\end{aligned}$$

$$(a^2 - ab + b^2)(a + b) = a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3$$
$$= a^3 + b^3.$$

65. When more than two polynomials are to be multiplied together, the result is obtained by successive operations. Thus, if $a-b$, a^2+b^2, $a+b$, have to be multiplied together, the product of the first two factors is

$$(a-b)(a^2+b^2) = a^3 - a^2b + ab^2 - b^3.$$

If this result is multiplied by the third factor,

$$\begin{aligned}(a-b)(a^2+b^2)(a+b) &= (a^3 - a^2b + ab^2 - b^3)(a+b) \\ &= a^4 - a^3b + a^2b^3 - ab^3 \\ &\quad + a^3b - a^2b^2 + ab^3 - b^4 \\ &= a^4 - b^4.\end{aligned}$$

66. When several factors have to be multiplied together, the process is often made less laborious if some judgment is shown in the order in which they are successively taken.

Ex. To multiply together $x+y$, x^4+y^4, $x-y$, x^2+y^2.

$$(x+y)\ (x-y) = x^2 - y^2 \ (64).$$
$$(x+y)(x-y)(x^2+y^2) = (x^2+y^2)(x^2-y^2) = x^4 - y^4.$$
$$(x+y)(x-y)(x^2+y^2)(x^4+y^4) = (x^4-y^4)(x^4+y^4) = x^8 - y^8.$$

67. If several factors multiplied together have a result o, one at least of the factors itself is o. For if all had finite values, their product would have a finite value. Therefore all have not finite values, and one factor at least is o, while it may be that more than one factor may be o.

68. If $(x-a)(x-b)=0$, then either $x-a$ or $x-b$ is o, and if a and b mean different finite quantities, the only inference to be drawn is that one of two quantities $x-a$, $x-b$ is o, the other being finite in value.

69. If $(x-a)(y-b)=0$, then it is certain that either $x-a$ is o, or $y-b$ is o, and it is possible that both $x-a$ and $y-b$ may each be o.

70. Ex. If $a^2-b^2=0$, either $a=b$ or $a=-b$ (64).

If $a^3-b^3=0$, „ $a=b$ or $a^2+ab+b^2=0$.

If $a^3+b^3=0$, „ $a=-b$ or $a^2-ab+b^2=0$.

Examples for Practice.

71. The following results can be verified as examples of multiplication:

1. $(x-2)(x+3)=x^2+x-6.$
2. $(a^2x-b^2)(a+bx)=a^3x+a^2bx^2-ab^2-b^3x.$
3. $x(mz-ny)+y(nx-bz)+z(by-mx)=0.$
4. $(3a-7x)(a+2x)=3a^2-ax-14x^2.$
5. $(a^2-ab+b^2)(a^4+a^3b-ab^3-b^4)=a^6-b^6$
6. $(a^3-2a^2b+3ab^2-4b^3)(a^2+4ab+4b^2)$
$=a^5+2a^4b-a^3b^2-4ab^4-16b^5.$
7. $(a^2-2a+5)(a+4)=a^3+2a^2-3a+20.$
8. $(m^2+2mn+n^2)(m^2-2mn+n^2)=m^4-2m^2n^2+n^4.$
9. $(1+x+x^2)(1-x+x^2)=1+x^2+x^4.$
10. $(a^4+a^2b^2+b^4)(a^4-a^2b^2+b^4)=a^8+a^4b^4+b^8.$
11. $(81x^4+27x^3y+9x^2y^2+3xy^3+y^4)(3x-y)=243x^5-y^5.$
12. $(2x^2-3xy-y^2)(2x^2+3xy-y^2)=4x^4-13x^2y^2+y^4.$
13. $(n^2-2nx+x^2)(n-x)=n^3-3n^2x+3nx^2-x^3.$
14. $(5x^2+x-3)(7x+10)=35x^3+57x^2-11x-30.$
15. $(7x^2-3x-9)(5x-4)=35x^2-43x^2-33x+36.$
16. $(x^4-2a^2x^2+a^4)(x^2-a^2)=x^6-3a^2x^4+3a^4x^2-a^6.$
17. $(x+\cdot 3)(x+\cdot 16)(x+1\cdot 25)=x^3+1\cdot 71x^2+\cdot 623x+\cdot 06.$
18. $(x^2+ax+b^2)(x^2+ax-b^2)=x^4+2a^2x^3+a^2x^2-b^4.$
19. $(a-b-c+d)(a+b-c-d)=a^2-b^2+c^2-d^2-2ac$
$+2bd.$
20. $(a+b+c)(bc+ac+ab)=a(b^2+bc+c^2)+b(c^2+ac+a^2)$
$+c(a^2+ab+b^2).$
21. $(a+b)(ax+b)(a-bx)=a^3x+a^2b+a^2bx-a^2bx^2+ab^2$
$-ab^2x^2-ab^2x-b^3x.$
22. $(x^2-4ax+a^2)(x-a)=x^3-5ax^2+5a^2x-a^3.$
23. $(x-3a)(x-a)(x+a)(x+3a)=x^4-10a^2x^2+9a^4.$
24. $(x-3)(x^2-2x+5)(x+3)=x^4-2x^3-4x^2+18x-45.$

25. $(a+b)(a-b)(a^2+ab+b^2)(a^2-ab+b^2)=a^6-b^6$ (64).

26. $(a^4+2a^3x+3a^2x^2+2ax^3+x^4)(a^4-2a^3x+3a^2x^2-2ax^3+x^4)$
$=a^8+2a^6x^2+3a^4x^4+2a^2x^6+x^8$.

27. $(a-b)(b-c)(c-a)+a^2(b-c)+b^2(c-a)+c^2(a-b)=0$.

28. $x(x+2)-x(x-2)=4x$.

29. $(x^2+x+1)y^2-(x^2-x+1)y^2=2xy^2$.

30. $(x^2+y^2)(x+y)-(x^2-y^2)(x-y)=2x^2y+2xy^2$
or $2xy(x+y)$.

31. $(a+c-b)(a+b-c)+(b+c-a)(b+a-c)$
$+(c+a-b)(c+b-a)=2(bc+ac+ab)-a^2-b^2-c^2$.

32. $(x+y+z)(yz+xz+xy)-xyz=(y+z)(z+x)(x+y)$.

33. $(x+a)(x-b)-(x-a)(x+b)=2(a-b)x$.

34. $(a-b)(a-b-x)(a+2b-2x)+b(b-x)(3a-2b-2x)$
$=a(a-x)(a-2x)$.

35. $(x^2-y^2-z^2)(2y^2-z^2)=(2y^2-z^2)x^2-2y^4-y^2z^2+z^4$.

36. $(x^2-y^2-z^2)(y^4-y^2z^2+z^4)=(y^4-y^2z^2+z^4)x^2$
$-y^6-z^6$.

37. $(x^2+y^2+z^2-yz-xz-xy)(x+y+z)$
$=x^3+y^3+z^3-3xyz$.

DIVISION.

72. The division of one algebraic quantity by another is expressed by writing the latter, which we will call, as in Arithmetic, the divisor, beneath the other, which we will call the dividend, the result signified being the quotient, and having the character, that this quotient and the divisor, when multiplied together, make the dividend (25).

73. If the divisor is a monomial, and the dividend a monomial also, any numerical coefficient in the divisor operates on a numerical coefficient in the dividend according to the rules of Arithmetic. Thus $\frac{6}{3}$ gives 2, $\frac{12}{2}$ gives 6, $\frac{8}{16}$ gives $\frac{1}{2}$. Where one number does not exactly divide the

other, they can only be left in their lowest terms, as, for instance, $\frac{56}{63}$ replaced by $\frac{8}{9}$.

When the same letter occurs in dividend and divisor, they operate upon one another in consistence with the general law for multiplication, $a^{m+n}=a^m \times a^n$ (56).

Hence, $$\frac{a^{m+n}}{a^n}=a^m,$$

inasmuch as a^m and a^n multiplied together give a^{m+n}.

So $$\frac{a^m}{a^m}=1.$$

Also, $$\frac{a^m}{a^{m+n}} = \frac{1}{a^n}, \text{ since } \frac{a^m}{a^{m+n}} = \frac{a^m}{a^m \times a^n} = \frac{1}{a^n}.$$

Thus, $$\frac{a^5}{a^2}=a^3, \ \frac{a^5}{a^8}=\frac{1}{a^3}, \ \frac{a}{a^4}=\frac{1}{a^3}.$$

In every such instance it will be seen that the quotient and divisor multiplied together make the dividend.

When different letters appear in the divisor and dividend, they can only be left with the division expressed. It could be effected only if the special values of the letters were assigned, to make a connection between them, which is not existing while the symbols remain general and open to receive any values.

These several remarks are brought into use in the following examples. The rule of signs, it will be remembered, fixes the algebraic sign of the quotient (25).

Examples:

1. $\dfrac{6a^2b}{2a} = 3ab.$

2. $\dfrac{-12a^3b^2c}{2a^2bc} = -6ab.$

3. $\dfrac{8x^8y^2z}{-16x^3y^4z^2} = -\dfrac{x^5}{2y^2z}.$

4. $\dfrac{-56xy^2z^8}{-63xy^3} = \dfrac{8z^8}{9y}.$

74. When the dividend is a polynomial, and the divisor still a monomial, the divisor operates separately on each term of the dividend, and the quotient is the result of these operations. For if $a+b+c$ is to be divided by a monomial d, and if $\frac{a}{d}=p$, $\frac{b}{d}=q$, $\frac{c}{d}=r$, then $p+q+r$ is the quotient required, since $d(p+q+r)=dp+dq+dr=a+b+c$.

$$\text{Thus,}\quad \frac{6a^3-4a^2b+3ab^2}{2a} = \frac{6a^3}{2a} - \frac{4a^2b}{2a} + \frac{3ab^2}{2a}$$

$$= 3a^2 - 2ab + \frac{3b^2}{2}.$$

$$\frac{15x^5+3x^2y-10xyz}{5x^2} = 3x^3 + \frac{3}{5}y - \frac{2yz}{x}.$$

75. When the dividend is the result of factors in brackets, the expressed multiplications are to be performed and the division thus effected as far as may be on each term.

Ex. 1. $\frac{(x^2-y^2)(x^2+y^2)}{xy} = \frac{x^4-y^4}{xy} = \frac{x^3}{y} - \frac{y^3}{x}.$

[*Obs.*—The product of the two factors which form the dividend in this example can at once be written down by (64) if x^2 is understood to be taking the place of a, and y^2 of b.]

Ex. 2. $\frac{(x^2+a)(x^4-ax^2+a^2)}{ax} = \frac{x^6-a^3}{ax}\ (64) = \frac{x^5}{a} - \frac{a^2}{x}.$

76. When the divisor is a polynomial, after the dividend has been cleared of all brackets, if any there be in it, the division is effected by a process similar to that of long division of numbers. The division of $a^2+2ab+b^2$ by $a+b$, will stand thus :

$$\begin{array}{rll}
a+b)\,a^2+2ab+b^2\,(a+b & & \\
\underline{a^2+\ ab\quad} & \quad\ldots & (1) \\
ab+b^2 & \quad\ldots & (2) \\
\underline{ab+b^2} & \quad\ldots & (3) \\
\underline{\cdot\ \ \cdot} & &
\end{array}$$

The first term of the dividend divided by the first term of the divisor suggests a as the first term of the quotient. The expression marked (1) results from multiplying the divisor by this first term. The expression (2) is the remainder when (1) is subtracted from the dividend. The first term of this remainder divided by the first term of the divisor suggests b as the next term of the quotient, and (3) is the result of multiplying the divisor by b. The fact of no remainder again arising shows the operation to be complete. The remainder, if there had been any, would be attached to the quotient in the form of an unaccomplished division.

Now this process, when it is analysed, will be found to be nothing more than a contrivance for arranging the dividend in parts on which the divisor is able to operate separately.

For the dividend

$$a^2+2ab+b^2=a^2+ab+ab+b^2$$
$$=a(a+b)+b(a+b).$$
$$\therefore \frac{a^2+2ab+b^2}{a+b}=\frac{a(a+b)}{a+b}+\frac{b(a+b)}{a+b}=a+b.$$

The term $a(a+b)$ is the quantity marked (1) in the preceding operation, and the other term is the quantity (2).

Ex. 1. To divide x^3-y^3 by $x-y$.

$$\begin{array}{rll}
x-y)\,x^3-y^3\ (x^2+xy+y^2 & & \\
x^3-x^2y & \quad .\quad .\quad . & (1) \\
\hline
x^2y-y^3 & & \\
x^2y-xy^2 & \quad .\quad .\quad . & (2) \\
\hline
xy^2-y^3 & & \\
xy^2-y^3 & \quad .\quad .\quad . & (3) \\
\hline
.\quad . & & \\
\hline
\end{array}$$

In this case the method is only equivalent to arranging the dividend in the manner following, where we employ the

artifice of adding and subtracting any the same quantity, without any influence upon value.

$$x^3-y^3=x^3-x^2y+x^2y-xy^2+xy^2-y^3$$
$$=x^2(x-y)+xy(x-y)+y^2(x-y).$$
$$\quad(1)\qquad\quad(2)\qquad\quad(3)$$
$$\therefore\ \frac{x^3-y^3}{x-y}=x^2+xy+y^2.$$

It will be observed that the terms marked (1) (2) (3) agree with those in the operation to which the same numbers are attached.

Ex. 2. To divide a^2+4b^2 by $a-2b$.

$$
\begin{array}{r}
a-2b)a^2+4b^2(a+2b\\
\underline{a^2-2ab\quad\ \ }\\
2ab+4b^2\\
\underline{2ab-4b^2}\\
8b^2
\end{array}
$$

$\therefore$ the quotient is $a+2b+\dfrac{8b^2}{a-2b}$.

The operation when analysed amounts to this:

$$a^2+4b^2=a^2-2ab+2ab+4b^2$$
$$=a^2-2ab+2ab-4b^2+8b^2$$
$$=a(a-2b)+2b(a-2b)+8b^2.$$
$$\therefore\ \frac{a^2+4b^2}{a-2b}=a+2b+\frac{8b^2}{a-2b}.$$

77. *Obs.*—In the division of one polynomial by another, according to the process just given, it is convenient that each be arranged in descending powers of the same letter. Thus, to divide $b^4-4ab^3-4a^3b+a^4+6a^2b^2$ by $3ab^2-b^3-3a^2b+a^3$, they would be arranged in the forms

$$\left.\begin{array}{l}b^4-4ab^3+6a^2b^2-4a^3b+a^4\\-b^3+3ab^2-3a^2b+a^3\end{array}\right\}$$

or

$$\left.\begin{array}{l}a^4-4a^3b+6a^2b^2-4ab^3+b^4\\a^3-3a^2b+3ab^2-b^3\end{array}\right\}$$

both being arranged in the former case by descending powers of b, in the latter by descending powers of a.

78. *Caution.*—Although multiplication by a polynomial is effected by performing the multiplication by each separate term of the multiplier and adding the results, division cannot be effected in like manner by dividing by each term of the divisor and adding the results. A numerical example will illustrate this. If 180 is to be multiplied by 6+4 or 10, it is correct to multiply 180 first by 6 and afterwards by 4 and to add the results, 1080 and 720 making 1800. But if 180 were to be divided by 6+4 or 10, it would not be correct to divide 180 first by 6 and then by 4 and to add the results, 30 and 45 not making 18.

79. Ex. 3. Divide $x^3+5a^2x-(a^3+5ax^2)$ by $x-a$.

The bracket is to be first removed, and the quantity to be divided arranged according to descending powers of some letter such as x, according to which the divisor is also arranged. Then the operation may stand thus :

$$\begin{array}{l} x-a)x^3-5ax^2+5a^2x-a^3(x^2-4ax+a^2 \\ \quad\;\; \underline{x^3-ax^2\qquad\qquad\quad} \\ \qquad\;\; -4ax^2+5a^2x-a^3 \\ \qquad\;\; \underline{-4ax^2+4a^2x\qquad} \\ \qquad\qquad\qquad\;\; a^2x-a^3 \\ \qquad\qquad\qquad\;\; \underline{a^2x-a^3} \\ \qquad\qquad\qquad\;\; \underline{\;\cdot\quad\cdot\quad\cdot\;} \end{array}$$

This division may be more easily performed if the formula (64) be kept in mind.

The quantity to be divided is

$$\begin{aligned} &x^3-a^3-5ax^2+5a^2x \\ &=x^3-a^3-5ax(x-a), \end{aligned}$$

$\therefore$ after it is divided by $x-a$ the quotient is

$$x^2+ax+a^2-5ax=x^2-4ax+a^2.$$

80. *Obs.*—Every example of multiplication (71) can make, if the student pleases, at least two examples of division, as the product can be divided by any one of the factors which combine to produce it. So, again, every example of division can be made an example of multiplication in reproducing the dividend by multiplying the divisor and quotient together.

81. To prove that

$$a\frac{b^3-c^3}{b-c}+b\frac{c^3-a^3}{c-a}+c\frac{a^3-b^3}{a-b}=(a+b+c)(bc+ac+ab).$$

$$a\frac{b^3-c^3}{b-c}+b\frac{c^3-a^3}{c-a}+c\frac{a^3-b^3}{a-b}$$

$$=a(b^2+bc+c^2)+b(c^2+ac+c^2)+c(a^2+ab+b^2)\ (64).$$

and this by a different arrangement

$$\begin{aligned}&=abc+a^2c+a^2b\\&+b^2c+abc+ab^2\\&+bc^2+ac^2+abc\\&=a(bc+ac+ab)+b(bc+ac+ab)+c(bc+ac+ab)\\&=(a+b+c)(bc+ac+ab).\end{aligned}$$

82. *Examples of Division for Practice.*

1. $(x^2-x-6)\div(x-3)=x+2.$
2. $(x^2-x-6)\div(x+2)=x-3.$
3. $(x^5-y^5)\div(x-y)=x^4+x^3y+x^2y^2+xy^3+y^4.$
4. $(b-3b^2+3b^3-b^4)\div(b-1)=-b+2b^2-b^3$ (77).
5. $(3a^2-13ax+14x^2)\div(3a-7x)=a-2x.$
6. $(x^4-10a^2x^2+9a^4)\div(x^2-9a^2)=x^2-a^2.$
7. $\{ab(x^2+y^2)+xy(a^2+b^2)\}\div(ax+by)=bx+ay.$
8. $(12a^3b^2+22b^3a^2+6ab^4)\div(3ba+b^2)=4a^2b+6ab^2.$
9. Divide $acr^3+(bc+ad)r^2+(bd+ae)r+be$ by $ar+b.$

The result $=cr^2+dr+e.$

10. $(x^6-3a^2x^4+3a^4x^2-a^6)\div(x^2-a^2)=x^4-2a^2x^2+a^4$.

11. $(a^2+ab+ac+bc)\div(a+c)=a+b$.

12. $(c^2+bc+ac+ab)\div(c+b)=a+c$.

13. $\{a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)\}\div(c-a)$
$=ab-ac-b^2+bc$.

14. $(x^4-a^2x^2-2ab^2x-b^4)\div(x^2+ax+b^2)=x^2-ax-b^2$.

15. $(x^4+2ax^3+a^2x^2-b^4)\div(x^2+ax-b^2)=x^2+ax+b^2$.

16. $(x^3+3ax^2+3a^2x+a^3)\div(x^2+2ax+a^2)=x+a$.

17. $(a^4+a^2x^2+x^4)\div(a^2+ax+x^2)=a^2-ax+x^2$.

18. Divide $p^2+pq+2pr-2q^2+7qr-3r^2$ by $p-q+3r$.
The result $=p+2q-r$

19. Divide $x^5+3x^4-5x^3-7x^2+12x-4$ by x^2+3x-2.
The result $=x^3-3x+2$.

20. $(a^6-2a^3b^3+b^6)\div(a^2-2ab+b^2)=a^4+2a^3b+3a^2b^2$
$+2ab^3+b^4$.

21. $(a^6+2a^3b^3+b^6)\div(a^2+2ab+b^2)=a^4-2a^3b+3a^2b^2$
$-2ab^3+b^4$

22. $(x^6-y^6)\div(x^2-xy+y^2)=x^4+x^3y-xy^3-y^4$.

23. $(x^6-2a^3x^3+a^6)\div(x^2+a^2-2ax)=x^4+2ax^3+3a^2x^2$
$+2a^3x+a^4$.

24. Divide $x^4+(2b^2-a^2)x^2+b^4$ by x^2+ax+b^2.
Result $=x^2-ax+b^2$.

25. $\{(2y^2-z^2)x^2-2y^4-y^2z^2+z^4\}\div(2y^2-z^2)=x^2-y^2-z^2$.

26. $\{(y^4-y^2z^2+z^4)x^2-y^6-z^6\}\div(x^2-y^2-z^2)$
$=y^4-y^2z^2+z^4$.

27. $(x^3+y^3+z^3-3xyz)\div(x+y+z)=x^2+y^2+z^2$
$-yz-xz-xy$.

28. $(a^5-3a^4b+a^3b^2-a^2b^3-ab^4-b^5)\div(a^3-2a^2b-2ab^2$
$-b^3)$ $=a^2-ab+b^2$.

29. Divide $a^8+a^6b^2+a^4b^4+a^2b^6+b^8$
by $a^4+a^3b+a^2b^2+ab^3+b^4$.
The result $=a^4-a^3b+a^2b^2-ab^3+b^4$.

30. $\{a^2-b^2+c^2-d^2-2(ac-bd)\}\div(a-b-c+d)$
$=a+b-c-d.$

31. Divide $a^2x^2+b^2y^2-(a^2b^2+x^2y^2)$ by $ax+by+ab+xy.$
The result $=ax+by-ab-xy.$

32. $(x^4+y^4-z^4+2x^2y^2-2z^2-1)\div(x^2+y^2-z^2-1)$
$=x^2+y^2+z^2+1.$

33. $\{x^4-2a(a-b)x^2+(a^2+b^2)(a-b)x-a^2b^2\}$
$\div(x^2-\overline{a-b}x+b^2)=x^2+\overline{a-b}x-a^2.$

34. Divide $x^6-na^2x^4+na^4x^2-a^6$ by $x^2-a^2.$
The result $=x^4+(1-n)a^2x^2+a^4.$

35. Multiply a^2-3a+2 by a^3-3a^2+2a, and divide the result by $a^2-2a+1.$
The result $=a^3-4a^2+4a.$

36. Multiply together $a-2$, $a-1$, $a+1$, $a+2$, and divide the result by $a^2-a-2.$
Answer $a^2+a-2.$

37. Multiply b^3+c^3 by b^3-c^3, and divide the product by $b^3-2b^2c+2bc^2-c^3.$
The result $=b^3+2b^2c+2bc^2+c^3.$

Involution.

83. If the reader is now qualified to multiply quantities together, involution will present no difficulty, being the multiplication of an algebraic quantity by itself as many times as the exponent (30) prescribes.

84. The following formulæ, which can be readily verified, deserve to be retained in the memory on account of their continual usefulness.

$$(a+b)^2=(a+b)\ (a+b)=a^2+2ab+b^2 \qquad \text{(i.)}$$
$$(a-b)^2=(a-b)\ (a-b)=a^2-2ab+b^2 \qquad \text{(ii.)}$$
$$(a+b)^3=(a+b)^2(a+b)=a^3+3a^2b+3ab^2+b^3. \qquad \text{(iii.)}$$
$$(a-b)^3=(a-b)^2(a-b)=a^3-3a^2b+3ab^2-b^3. \qquad \text{(iv.)}$$

85. *Caution.*—$(a+b)^2$ is not equivalent to a^2+b^2.

The first results from adding a to b and squaring the sum.

The second results from separately squaring a and b and adding together the squares.

They differ by the quantity $2ab$.

For instance, if 4 and 3 be the particular values of a and b, $(a+b)^2$ is the square of 7 or 49.

a^2+b^2 is 16 and 9, or 25.

86. By means of these useful formulæ (84) the square or cube of any binomial can at once be written down.

Ex. 1. $(5x-y)^2$.

If we regard $5x$ as occupying the place of a in ii. and y as occupying the place of b,

$$\begin{aligned}(5x-y)^2 &= (5x)^2-2\times 5x \,.\, y+y^2 \text{ by ii.}\\ &= 25x^2-10xy+y^2.\end{aligned}$$

2. $(x^2-2x)^2=x^4-4x^3+4x^2$.

3. $(x^2-4x)^3=(x^2)^3-3(x^2)^2 4x+3x^2(4x)^2-(4x)^3$ by iv.
$$=x^6-12x^5+48x^4-64x^3.$$

4. $(x+3a)^2=x^2+2x\times 3a+9a^2=x^2+6ax+9a^2$.

5. $(4ab+x)^3=(4ab)^3+3\times(4ab)^2x+3\times 4abx^2+x^3$
$$=64a^3b^3+48a^2b^2x+12abx^2+x^3.$$

87. The same formulæ may be made available to present, though not so readily, the squares and cubes of multinomials. They must first, by means of brackets, be regarded as binomials. For example,

$$(x^2-2x+1)^2=\{(x^2-2x)+1\}^2.$$

If in i. x^2-2x be regarded as standing in the place of a, and 1 as standing in the place of b,

$$\begin{aligned}(x^2-2x+1)^2 &= (x^2-2x)^2+2(x^2-2x)+1,\\ &= x^4-4x^3+4x^2+2x^2-4x+1,\\ &= x^4-4x^3+6x^2-4x+1.\end{aligned}$$

Ex. 2. $(a+b+c+d)^3=(\overline{a+b}+\overline{c+d})^3$

$$\begin{aligned}
&=(a+b)^3+3(a+b)^2(c+d)\\
&\qquad +3(a+b)(c+d)^2+(c+d)^3.\\
&=a^3+3a^2b+3ab^2+b^3\\
&+3(a^2+2ab+b^2)(c+d)\\
&+3(a+b)(c^2+2cd+d^2)\\
&+c^3+3c^2d+3cd^2+d^3,\\
&=a^3+b^3+c^3+d^3\\
&+3(a^2b+a^2c+a^2d+b^2a+b^2c+b^2d\\
&+c^2a+c^2b+c^2d+d^2a+d^2b+d^2c)\\
&+6(dbc+acd+abd+abc).
\end{aligned}$$

88. *Examples of Involution for Practice.*

1. $(x+3)=x^2+6x+9.$
2. $(4x-5)^2=16x^2-40x+25.$
3. $(3x-1)^2=9x^2-6x+1.$
4. $(4x-2)^2=16x^2-16x+4.$
5. $(5x-3)^2=25x^2-30x+9.$
6. $(x^2-y^2)^3=x^6-3x^4y^2+3x^2y^4-y^6.$
7. $(a^2-x^2)^3=a^6-3a^4x^2+3a^2x^4-x^6.$
8. $(a+b)^2+(a-b)^2=2(a^2+b^2).$
9. $(a+b)^2-(a-b)^2=4ab.$
10. Prove that
    $(ac-bd)^2+(ad+bc)^2=(a^2+b^2)(c^2+d^2).$
11. What is the coefficient of x in the expression $(a-b)^2x-(a-bx)^2$? *Ans.* a^2+b^2.
12. $(x-2)(x-4)(x-6)-3(x-2)^2=x^3-15x^2+56x-60.$
13. $\{(3x+2y)^2-(2x-3y)^2\}\div(x+5y)=5x-y.$
14. $\{(3x-2y)^2-(2x+y)^2\}\div(5x-y)=x-3y.$
15. $(x+\cdot 5)^2(x+\cdot 75)=x^3+1\cdot 75x^2+x+\cdot 1875.$
16. $(x-3a)^2-(x-2a)^2=5a^2-2ax.$
17. $(2x+a)^3-(x+a)^3=7x^3+9x^2a+3xa^2.$
18. $(a^2+ab+b^2)^2-(a^2-ab+b^2)^2=4ab(a^2+b^2).$

19. Prove that

$$(x+y+z)^2+x^2+y^2+z^2=(z+y)^2+(x+z)^2+(y+x)^2.$$

20. $(x+y+z)^3=x^3+y^3+z^3+3(y+z)(z+x)(x+y)$.

21. Simplify $x^2-[(x-y)^2-\{(x-y-z)^2-(z-x)^2\}]$.

Ans. $2yz$.

22. If $2a=x+y+z$

$$2b=-x+y+z$$
$$2c=x-y+z$$
$$2d=x+y-z,$$

then $a^2+b^2+c^2+d^2=x^2+y^2+z^2$.

89. When the expression to the squared is itself the square root of some quantity, the operation of squaring merely delivers this quantity from the square root. Thus,

$$(\sqrt{a+b})^2=a+b.$$

For by its definition the square root of a quantity, such as the square root of $a+b$, means such a quantity as multiplied by itself, or squared, produces $a+b$ (32).

Hence, $(\sqrt{a+b})^3=\sqrt{a+b}\;\sqrt{a+b}\;\sqrt{a+b}=(a+b)\sqrt{a+b}$,

and

$$\begin{aligned}(\sqrt{a+b})^4&=\sqrt{a+b}\;\sqrt{a+b}\;\sqrt{a+b}\;\sqrt{a+b}\\&=(a+b)(a+b)\\&=a^2+2ab+b^2.\end{aligned}$$

90. Ex. To simplify the expression

$$(a+b-c)^2(a-b+c)+(a+b+c)(a+b-c)(b+c-a).$$

It will be observed that in each of the two terms of which this expression consists the factor $a+b-c$ enters.

Hence the expression takes the form

$$\begin{aligned}&(a+b-c)\{(a+b-c)(a-b+c)+(a+b+c)(b+c-a)\}\\&=(a+b-c)\{(a+\overline{b-c})(a-\overline{b-c})+(\overline{b+c}+a)(\overline{b+c}-a)\}\\&=(a+b-c)\{a^2-(b-c)^2+(b+c)^2-a^2\}\ (84)\\&=(a+b-c)\{(b+c)^2-(b-c)^2\}\\&=4(a+b-c)bc\ (84).\end{aligned}$$

EVOLUTION.

91. Extracting a square root of a proposed algebraical quantity is the finding another quantity which, when squared, or multiplied by itself, will produce the former quantity.

92. If it is a monomial whose square root is to be found, a square root of each factor of it is taken or expressed, and the product of these square roots is a square root of the monomial proposed. Thus a square root of $16a^2b^4$ is $4ab^2$, because $4ab^2 \times 4ab^2 = 16a^2b^4$. A square root of $5a^2b$ is $\sqrt{5} \times a\sqrt{b}$ or $a\sqrt{5b}$, since $a\sqrt{5b} \times a\sqrt{5b} = 5a^2b$.

93. Since either a or $-a$ squared produces a^2, a or $-a$ is a square root of a^2. This result is expressed by stating that $\pm a$ is the square root of a^2, the sign $\pm$ meaning that either the positive or negative sign may be employed. When a square root of an expression has been found, the same quantity with the contrary sign will also be a square root of the same expression. Then as we have seen that $4ab^2$ is a square root of $16a^2b^4$, so is $-4ab^2$, and generally $\sqrt{16a^2b^4} = \pm 4ab^2$.

Similarly, there will be two fourth roots of the same quantity contrary in sign.

94. Square roots of quantities are multiplied together by multiplying the quantities and placing the result of that multiplication under the sign of the square root: thus, $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$. For $\sqrt{ab}$ means a quantity which, multiplied by itself makes ab. Now if $\sqrt{a}\ \sqrt{b}$ be multiplied by itself, it gives $\sqrt{a} \times \sqrt{a} \times \sqrt{b} \times \sqrt{b}$ or $a \times b$. Wherefore $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$.

Ex. 1. $\sqrt{8a^2b} = \pm\sqrt{8} \,.\, a\sqrt{b} = \pm\sqrt{4}\sqrt{2}a\sqrt{b}$
$= \pm 2\sqrt{2} \,.\, a\sqrt{b} = \pm 2a\sqrt{2b}$.

Ex. 2. $5\sqrt{54x^3y^2z} = 5\sqrt{6}\sqrt{9}\sqrt{x^2} \,.\, x\sqrt{y^2}\sqrt{z}$
$= \pm 5\sqrt{6} \times 3x\sqrt{x} \,.\, y\sqrt{z}$
$= \pm 15xy\sqrt{6xz}$.

95. The cube, fourth, or higher roots of monomials may be similarly obtained.

Thus,

$$\sqrt[3]{a^3b^2c} = \sqrt[3]{a^3}\sqrt[3]{b^2c} = a\sqrt[3]{b^2c}.$$
$$\sqrt[3]{27xyz^4} = \sqrt[3]{27}\sqrt[3]{xyz}\sqrt[3]{z^3}$$
$$= 3\sqrt[3]{xyz}.z = 3z\sqrt[3]{xyz}.$$
$$\sqrt[4]{a^7b^5c^4} = \sqrt[4]{a^4}\sqrt[4]{a^3}\sqrt[4]{b^4}\sqrt[4]{b}\sqrt[4]{c^4}.$$
$$= \pm a\sqrt[4]{a^3}\, b\sqrt[4]{b}\, c.$$
$$= \pm abc\sqrt[4]{a^3b}.$$

96. The square root of binomials or polynomials can be obtained by a general method, which is too complicated to admit of explanation here. Practically, when the square root of a binomial can be obtained, the formulæ of (84) lead to it. By these

$$\sqrt{a^2+2ab+b^2} = \pm(a+b)$$
$$\sqrt{a^2-2ab+b^2} = \pm(a-b).$$

The reason of the double sign is given in (93).

Whenever an expression can be arranged in three terms, whereof the second, as to value without respect to sign, is twice the product of the square roots of the other two, then the square root of the expression can at once be written down from the formulæ just given.

Ex. 1. $25a^2+10ab+b^2$.

Here $5a$ and b being square roots of $25a^2$ and b^2, the middle term is double their product,

$$\therefore \quad \sqrt{25a^2+10ab+b^2} = \pm(5a+b).$$
$$\text{So} \quad \sqrt{25a^2-10ab+b^2} = \pm(5a-b).$$

Ex. 2. $\sqrt{a^4+2a^3+a^2} = \pm(a^2+a)$.

Ex. 3. $a^4+2a^3+3a^2+2a+1 = a^4+2a^3+a^2+2a^2+2a+1$
$$= (a^2+a)^2+2(a^2+a)+1$$
$$= (a^2+a+1)^2$$

$$\therefore \quad \sqrt{a^4+2a^3+3a^2+2a+1} = \pm(a^2+a+1).$$

Ex. 4. To find the square root of

$$4x^2y^4-12x^3y^3+17x^4y^2-12x^5y+4x^6.$$

$$\begin{aligned}4x^2y^4-12x^3y^3+17x^4y^2-12x^5y+4x^6\\ =4x^2y^4-12x^3y^3+9x^4y^2+8x^4y^2-12x^5y+4x^6\\ =(2xy^2-3x^2y)^2+2(2xy^2-3x^2y)\times 2x^3+(2x^3)^2\end{aligned}$$

$\therefore$ the square root required is

$$\pm(2xy^2-3x^2y+2x^3).$$

Ex. 5. $\sqrt{x^4+4x^3+6x^2+4x+1}=\pm(x^2+2x+1)$.

97. When the cube roots of polynomials admit of extraction, it will generally be dependent on the agreement of these polynomials with the form for $(a\pm b)^3$; namely, $a^3\pm 3a^2b+3ab^2\pm b^3$.

Ex. $x^3+6x^2y+12xy^2+8y^3$

$$\begin{aligned}&=x^3+3\times x^2\times 2y+3\times x\times(2y)^2+(2y)^3\\ &=(x+2y)^3.\end{aligned}$$

$\therefore\ \sqrt[3]{x^3+6x^2y+12xy^2+8y^3}=x+2y$.

Algebraical Fractions.

98. It has been seen that when the division of one quantity by another cannot be effected, it is left expressed in the form of a fraction. Thus, $\frac{a}{b}$ means a quantity which multiplied by b produces a, and if such a quantity cannot be found as long as a and b remain open to admit any values, and are therefore unconnected, the quantity must remain in the form $\frac{a}{b}$ and makes an algebraical fraction, a being called the numerator and b the denominator.

It will be observed that when a and b are positive integers, this conception of a fraction includes the usual arithmetical view of a fraction.

99. A fraction $\frac{a}{b}$ then means such a quantity as when multiplied by b will produce the result a.

If we describe the fraction by the symbol p, we mean that p multiplied by b must make a. In reasoning upon $\frac{a}{b}$ as the description of a fraction, a and b may be any algebraical quantities, resulting, it may be, from the multiplication of several factors.

100. If $\frac{a}{b}$ exceeds 1, a is greater than b.

If $\frac{a}{b}$ is less than 1, a is less than b.

If $\frac{a}{b}$ is equal to 1, a is equal to b.

101. A fraction is not altered in value if its numerator and denominator are both multiplied by the same quantity. This amounts to nothing more than increasing the magnitude to be divided in the same degree as the divisor is increased. For if $\frac{a}{b}$ is a certain quantity, this quantity multiplied by b makes a. Therefore the same quantity multiplied by cb makes ca; i.e. it is still the value of the fraction $\frac{ca}{cb}$.

102. By this principle fractions with different denominators can be altered in form without being altered in value, so as all to have the same denominator. Usually a quantity can be found by inspection which includes as factors their various denominators, and this quantity they can all be made to take as their denominator, by multiplying the numerator and denominator of each by some properly chosen multiplier.

Suppose the fraction be $\frac{1}{3a^2b}$, $\frac{c}{2ab^2}$, $\frac{d}{ab^3}$. The denomi-

nators $3a^2b$, $2ab^2$, ab^3 are all included in the quantity $6a^2b^3$. It will be our object therefore so to multiply the numerator and denominator of each fraction that $6a^2b^3$ may become the denominator of each.

$$\text{Now} \quad \frac{1}{3a^2b} = \frac{2b^2}{3a^2b \times 2b^2} = \frac{2b^2}{6a^2b^3},$$

$$\frac{c}{2ab^2} = \frac{c \times 3ab}{2ab^2 \times 3ab} = \frac{3abc}{6a^2b^3},$$

$$\frac{d}{ab^3} = \frac{d \times 6a}{ab^3 \times 6a} = \frac{6ad}{6a^2b^3};$$

and thus the fractions, without any alteration of their several values, have been made to have the same denominator.

Let the fractions be

$$\frac{3}{a-b}, \quad \frac{2}{a+b}, \quad \frac{a}{a^2-b^2}.$$

Since a^2-b^2 is the product of $a+b$ and $a-b$ (64), and thus includes them both, it is the denominator which we shall aim to make the fractions assume.

$$\frac{3}{a-b} = \frac{3(a+b)}{(a+b)(a-b)} = \frac{3a+3b}{a^2-b^2},$$

$$\frac{2}{a+b} = \frac{2(a-b)}{(a-b)(a+b)} = \frac{2a-2b}{a^2-b^2},$$

$$\frac{a}{a^2-b^2} = \frac{a}{a^2-b^2},$$

and the three fractions now have the same denominator.

103. If a fraction may have its numerator and denominator multiplied by any the same quantity without its value being altered, so also its numerator and denominator may be divided by any the same quantity, or have any common factor cancelled or expunged; because the fraction resulting after this division can be restored to the former value by the numerator and denominator being both multiplied by the cancelled factor.

Hence fractions can often be simplified, or made to take a more brief form than that in which they are appearing, if any factor can be found common to numerator and denominator, which may therefore be removed without altering the value of the fraction.

Ex. 1. $$\frac{15x^3y}{3xy^2} = \frac{5x^2}{y}.$$

Ex. 2. $$\frac{24x^my^3z}{15x^2y^nz} = \frac{8x^{m-2}}{5y^{n-3}}.$$

Ex. 3. In the fraction $\frac{x^2-4ax-5a^2}{x^2-6ax+5a^2}$, it will be found that both numerator and denominator are divisible by $x-5a$, and when this factor is removed the fraction takes the simpler equivalent form $\frac{x+a}{x-a}$.

104. *Caution.*—The most important reduction of fractions to simpler forms is usually accomplished by removing from numerator and denominator common factors, but it is to be noted that the whole of the numerator and the whole of the denominator must be divisible by the factor which is thus cancelled. For example:

$$\frac{x^2+3(x-3)}{2(x-3)} \text{ is not reducible to } \frac{x^2+3}{2},$$

because though one term of the numerator is divisible by $x-3$, the whole of the numerator is not divisible by this factor. The numerator and denominator indeed have no common factor. If the fraction had been $\frac{x^2-9+3(x-3)}{2(x-3)}$, then the numerator is divisible by $x-3$ and the fraction is reducible to the form $\frac{x+6}{2}$.

105. The following are examples of fractions simplified by cancelling or suppressing factors common to numerator and denominator.

1. $\dfrac{3x^3-2xy^2}{2x^2-3xy} = \dfrac{3x^2-2y^2}{2x-3y}.$

2. $\dfrac{a^2-ab-2b^2}{a^2+3ab+2b^2} = \dfrac{(a-2b)(a+b)}{(a+2b)(a+b)} = \dfrac{a-2b}{a+2b},$

the factor cancelled being $a+b$.

3. $\dfrac{(3a-2b)^2-(a+5b)^2}{(2a-7b)^2} = \dfrac{4a+3b}{2a-7b}.$

4. $\dfrac{(a+b+c)^2-d^2}{(a+b+c+d)^2} = \dfrac{a+b+c-d}{a+b+c+d}.$

106. If two fractions have the same denominator, they are added together by adding their numerators to make the numerator of a new fraction with the denominator aforesaid. If the fractions be $\frac{a}{d}$ and $\frac{b}{d}$, these are quantities which multiplied severally by d give a and b as the results (99). Hence, the fractions added together and multiplied by d make $a+b$. The sum of the fractions therefore is the fraction $\frac{a+b}{d}$ (99).

This reasoning can be extended to any number of fractions which have the same denominators, for when two are added together, like those just considered, and combined into one, another with the same denominator can be united to them by the same rule, and so the process can be carried through any number of such fractions.

107. Similarly, one fraction can be subtracted from another if they have the same denominator. For, as before, one fraction multiplied by d makes a, and the other multiplied by d makes b; therefore their difference multiplied by d makes $a-b$, or their difference is the fraction $\frac{a-b}{d}$.

108. *Obs.*—If any difficulty be found in following the reasoning upon algebraic fraction, it often disappears if the same reasoning is used in particular cases, which may be formed at pleasure by giving numerical values to the symbols.

109. If fractions are to be added together, or one subtracted from others, when they have not the same denominator, let them be altered in form without being altered in value, so that they may all have the same denominator (102). Then the method of the preceding article applies to them, and they are united by the addition or subtraction of the numerators which they have in their altered state.

Ex. 1. Let $\frac{x}{2x-1}$, $\frac{x}{2x+1}$, and $\frac{1}{1-4x^2}$ be added together.

The quantity $4x^2-1$ will include all the denominators, since it is $(2x-1)(2x+1)$,

$$\frac{x}{2x-1} = \frac{x(2x+1)}{4x^2-1} = \frac{2x^2+x}{4x^2-1},$$

$$\frac{x}{2x+1} = \frac{x(2x-1)}{4x^2-1} = \frac{2x^2-x}{4x^2-1},$$

$$\frac{1}{1-4x^2} = -\frac{1}{4x^2-1} \text{ (42. } Obs.\text{)},$$

$$\therefore \text{ Sum } = \frac{4x^2-1}{4x^2-1} = 1.$$

Ex. 2. To simplify $\frac{1+x}{1+x+x^2} - \frac{1-x}{1-x+x^2}$.

The denominator to be taken as a common denominator must be the product of the two denominators, viz. $1+x^2+x^4$.

$$\frac{1+x}{1+x+x^2} = \frac{(1+x)(1-x+x^2)}{1+x^2+x^4}$$

$$= \frac{1+x^3}{1+x^2+x^4} \text{ (64)},$$

$$-\frac{1-x}{1-x+x^2} = -\frac{(1-x)(1+x+x^2)}{1+x^2+x^4}$$

$$= -\frac{1-x^3}{1+x^2+x^4} \text{ (64)}.$$

$$\therefore \frac{1+x}{1+x+x^2} - \frac{1-x}{1-x+x^2} = \frac{2x^3}{1+x^2+x^4}.$$

Ex. 3. $\dfrac{a^2+b^2}{a^2-b^2} - \dfrac{a-b}{a+b}$.

Since a^2-b^2 includes $a+b$ it may be taken as the common denominator.

$$\frac{a^2+b^2}{a^2-b^2} = \frac{a^2+b^2}{a^2-b^2},$$

$$-\frac{a-b}{a+b} = -\frac{a^2-2ab+b^2}{a^2-b^2},$$

$$\therefore \frac{a^2+b^2}{a^2-b^2} - \frac{a-b}{a+b} = \frac{2ab}{a^2-b^2}.$$

Ex. 4. From x^3+3 subtract $\dfrac{3x-5}{x-2}$, so as to express the difference by a single fraction. To effect this the quantity x^3+3 must be made to take the form of a fraction whose denominator is $x-2$. In this form it is

$$\frac{(x^3+3)(x-2)}{x-2} = \frac{x^4-2x^3+3x-6}{x-2}.$$

$\therefore$ the result of subtraction required is

$$\frac{x^4-2x^3+3x-6}{x-2} - \frac{3x-5}{x-2}$$

$$= \frac{x^4-2x^3-1}{x-2}.$$

Ex. 5. To add together the fractions

$$\frac{x^2y^2}{(z^2-x^2)(z^2-y^2)},\quad \frac{x^2z^2}{(y^2-x^2)(y^2-z^2)},\quad \frac{y^2z^2}{(x^2-y^2)(x^2-z^2)}.$$

A denominator, including the denominators of these three fractions, is observed to be

$$(x^2-y^2)(y^2-z^2)(z^2-x^2),$$

and to this common denominator they will be reduced.

$$\frac{x^2y^2}{(z^2-x^2)(z^2-y^2)} = -\frac{x^2y^2}{(z^2-x^2)(y^2-z^2)} \text{ (42. } Obs.)$$

$$= -\frac{x^2y^2(x^2-y^2)}{(x^2-y^2)(y^2-z^2)(z^2-x^2)},$$

$$\frac{x^2z^2}{(y^2-x^2)(y^2-z^2)} = -\frac{x^2z^2(z^2-x^2)}{(x^2-y^2)(y^2-z^2)(z^2-x^2},$$

$$\frac{y^2z^2}{(x^2-y^2)(x^2-z^2)} = -\frac{y^2z^2(y^2-z^2)}{(x^2-y^2)(y-z^2)(z^2-x^2)},$$

$\therefore$ the numerators together give

$$x^2y^2(y^2-x^2)+x^2z^2(x^2-z^2)+y^2z^2(z^2-y^2)$$
$$=x^4z^2-x^4y^2+x^2y^4-x^2z^4+y^2z^2(z^2-y^2)$$
$$=x^4(z^2-y^2)-x^2(z^4-y^4)+y^2z^2(z^2-y^2)$$
$$=(z^2-y^2)\{x^4-x^2z^2-x^2y^2+y^2z^2\}$$
$$=(z-y^2)(x^2-y^2)(x^2-z^2)$$
$$=(x^2-y^2)(y^2-z^2)(z^2-x^2),$$

$\therefore$ the fractions give as their sum

$$\frac{(x^2-y^2)(y^2-z^2)(z^2-x^2)}{(x^2-y^2)(y^2-z^2)(z^2-x^2)}, \text{ or unity.}$$

110. Fractions to be combined by addition or subtraction are generally to be simplified as far as possible before they are brought to a common denominator.

Ex. 1. To find the difference of $\frac{x^2+2x+2}{x+1}$ and $\frac{x^2+4x+6}{x+2}$

$$\frac{x^2+2x+2}{x+1} = x+1 + \frac{1}{x+1} = x+1 + \frac{x+2}{(x+1)(x+2)}$$

$$\frac{x^2+4x+6}{x+2} = x+2 + \frac{2}{x+2} = x+2 + \frac{2(x+1)}{(x+1)(x+2)}$$

$$\therefore \quad \frac{x^2+4x+6}{x+2} - \frac{x^2+2x+2}{x+1} = 1+\frac{x}{(x+1)(x+2)}.$$

Ex. 2. $\frac{x^2+8x+20}{x+4} - \frac{x^2+6x+12}{x+3} = 1+\frac{x}{(x+3)(x+4)}.$

111. *Examples for Practice in Addition and Subtraction of Fractions.*

1. $\dfrac{3}{x-a} + \dfrac{5}{x+a} = \dfrac{8x-2a}{x^2-a^2}$ or $2\dfrac{4x-a}{x^2-a^2}$.

2. $\dfrac{a^3-b^3}{a-b} - \dfrac{a^3+b^3}{a+b} = 2ab.$

3. $\dfrac{1}{b+c} - \dfrac{1}{c+a} = \dfrac{a-b}{bc+ac+c^2+ab}.$

4. $\dfrac{1}{c+a} - \dfrac{1}{a+b} = \dfrac{b-c}{a^2+ac+bc+ab}.$

5. $\dfrac{2x}{x^2-a^2} - \dfrac{1}{x-a} = \dfrac{1}{x+a}.$

6. $\dfrac{a}{b} - \dfrac{ac}{b(b+c)} = \dfrac{a}{b+c}.$

7. $\dfrac{5x-9}{x+3} + \dfrac{2x+11}{x-2} = \dfrac{7x^2-2x+51}{x^2+x-6}.$

8. $\dfrac{5x}{x-2} - \dfrac{2(5x+1)}{x^2-4} = \dfrac{5x^2-2}{x^2-4}.$

9. $\dfrac{a}{a-2} + \dfrac{a}{a-3} = \dfrac{2a^2-5a}{a^2-5a+6}.$

10. $\dfrac{a}{a-2} - \dfrac{a}{a-3} = -\dfrac{a}{a^2-5a+6}.$

11. $\dfrac{x-1}{5} - \dfrac{x-11}{7} + \dfrac{3x-(5x-4)}{2} = 3\dfrac{13}{35} - \dfrac{33x}{35}.$

12. $\dfrac{a}{b} + \dfrac{c}{b} \cdot \dfrac{bd-ae}{ce-bf} = \dfrac{bcd-abf}{b(ce-bf)} = \dfrac{cd-af}{ce-bf}.$

13. $\dfrac{1}{abx} + \dfrac{1}{a(a-b)(x-a)} + \dfrac{1}{b(b-a)(x-b)}$

$$= \frac{1}{x(x-a)(x-b)}.$$

14. $\dfrac{x^3}{x-1} - \dfrac{x^2}{x+1} - \dfrac{1}{x-1} + \dfrac{1}{x+1} = x^2+2.$

15. $\dfrac{x^2+2x+3}{x^3-1} + \dfrac{x+2}{x^2+x+1} + \dfrac{1}{x-1}$
$= \dfrac{3x^2+4x+2}{x^3-1}$ (64).

16. $\dfrac{x}{x^2+y^2} + \dfrac{x}{x^2-y^2} + \dfrac{y^2}{(x-y)(x^2+y^2)}$
$- \dfrac{2x^3-y^3-xy^2}{x^4-y^4} = 2\,\dfrac{y^3+xy^2}{x^4-y^4}.$

17. $\dfrac{1}{x-1} - \dfrac{1}{2x+2} - \dfrac{x-3}{2x^2+2} = \dfrac{3x^2+x}{x^4-1}.$

18. $\dfrac{a^2}{a^2-1} + \dfrac{a}{a-1} + \dfrac{a}{a+1} = \dfrac{3a^2}{a^2-1}.$

19. $\dfrac{1}{3x-1} + \dfrac{2}{x-1} - \dfrac{1}{x} = \dfrac{4x^2+x-1}{(3x-1)(x-1)x}.$

20. $\dfrac{1}{x-1} - \dfrac{1}{2x+2} - \dfrac{x+3}{2x^2+2} = \dfrac{x+3}{x^4-1}.$

21. $\dfrac{1}{3-x^2} + \dfrac{1}{3+x^2} + \dfrac{1}{x^4-9} = \dfrac{5}{9-x^4}.$

22. Add together $\dfrac{a(a+3)}{(a+1)(a+2)}$ and $\dfrac{2}{3a(a+2)}$, and find the value of the result when $a = \cdot 5$. The result is 1.

112. If several fractions equal to one another be taken, as for instance $\frac{2}{3}$, $\frac{10}{15}$, $\frac{14}{21}$, $\frac{4}{6}$, it will be found that if their numerators be added together to make a new numerator, and their denominators added together to make a new denominator, the fraction resulting, $\frac{30}{45}$, is equal to any one of the former fractions. By the application of algebra it can thus be shown that this is the case with any system of equal fractions, whatever their number may be.

Let $\frac{a_1}{b_1}$, $\frac{a_2}{b_2}$, ... $\frac{a}{b_n}$ be n fractions equal to one another. Let k represent the value of each of them:

$$\therefore \frac{a_1}{b_1}=k, \qquad a_1=b_1k,$$
$$\frac{a_2}{b_2}=k, \qquad a_2=b_2k,$$
$$. \quad . \quad . \quad .$$
$$\frac{a_n}{b_n}=k, \qquad a_n=b_nk.$$

$$\therefore a_1+a_2+a_3+\ldots+a_n=b_1k+b_2k+\ldots+b_nk$$
$$=k(b_1+b_2+\ldots+b_n)$$

$\therefore \frac{a_1+a_2+a_3+\ldots+a_n}{b_1+b_2+\ldots \quad +b^n}=k$, which is the value of any one of the original fractions.

Multiplication of Fractions.

113. Let $\frac{a}{b}$ be a fraction which may be denoted by p and $\frac{c}{d}$ another denoted by q. The meaning then is that p multiplied by b makes a, and q multiplied by d makes c (98). Therefore if p, q, b, d be multiplied together, the result is ac. But b and d multiplied together give bd. $\therefore$ the product of p and q multiplied by bd is ac. According then to the definition the product of p and q is the fraction $\frac{ac}{bd}$.

In other words, two fractions are multiplied together by multiplying their numerators together to make the numerator of the result, and their denominators together to make the denominator of the result.

114. If a fraction $\frac{a}{b}$ is multiplied by $\frac{b}{a}$ the result is unity. For the fraction $\frac{ab}{ab}$, meaning a quantity which when multiplied by ab makes ab, is unity.

115. It has been already explained (42) that if the signs of both the numerator and the denominator of a fraction are reversed, the fraction is not altered; but if the sign of either the numerator or the denominator be reversed, the sign of the fraction is reversed.

$$\text{Thus, } \frac{-a}{-b}=\frac{a}{b}, \text{ but } \frac{-a}{b}=-\frac{a}{b}, \frac{a}{-b}=-\frac{a}{b}.$$

DIVISION OF FRACTIONS.

116. If a fraction $\frac{a}{b}$ is to be divided by a fraction $\frac{c}{d}$, a result is required such that when it is multiplied by $\frac{c}{d}$ it shall make $\frac{a}{b}$. After being thus multiplied, suppose that the result were furthermore multiplied by $\frac{d}{c}$. It must produce $\frac{ad}{bc}$ (113). But now the multiplier of the result is $\frac{c}{d}\cdot\frac{d}{c}$ or unity (114). Hence the result of dividing $\frac{a}{b}$ by $\frac{c}{d}$ is $\frac{ad}{bc}$.

A fraction then is divided by another by inverting the latter, and multiplying the former by the fraction produced by this inversion.

It will be observed how these rules of operation in algebraic fractions agree with the rules by which arithmetical fractions are combined.

117. If two expressions are to be combined by multiplication or division, and one or both of them is the sum or difference of fractions, these expressions may have to be prepared before they can be brought under the rules just given, and be reduced each to a single fraction by the method of (109).

Let it be required, for example, to divide

$$\frac{x}{1+x} + \frac{1-x}{x} \text{ by } \frac{x}{1+x} - \frac{1-x}{x}.$$

Now $\dfrac{x}{1+x} + \dfrac{1-x}{x} = \dfrac{x^2}{x(1+x)} + \dfrac{1-x^2}{x(1+x)} = \dfrac{1}{x(1+x)}$,

$$\frac{x}{1+x} - \frac{1-x}{x} = \frac{x^2}{x(1+x)} - \frac{(1-x^2)}{x(1+x)} = \frac{2x^2-1}{x(1+x)}.$$

Hence the former divided by the latter gives

$$\frac{1}{x(1+x)} \times \frac{x(1+x)}{2x^2-1)} = \frac{1}{2x^2-1}.$$

118. ***Examples for Practice in Multiplication and Division of Fractions.***

1. $\dfrac{a^2-9}{ab} \times \dfrac{b^2}{a+3} = \dfrac{(a+3)b}{a} = b + \dfrac{3b}{a}$.

2. $\dfrac{a^3-b^3}{a^3+b^3} \times \dfrac{a+b}{a-b} = \dfrac{a^2+ab+b^2}{a^2-ab+b^2} = 1 + \dfrac{2ab}{a^2-ab+b^2}$.

3. $\dfrac{x-2}{x-3} \times \dfrac{x+3}{x+4} = \dfrac{x^2+x-6}{x^2+x-12} = 1 + \dfrac{6}{x^2+x-12}$.

4. Multiply $\dfrac{x^2+xy}{x^2+y^2}$ by $\dfrac{x}{x-y}$ and by $\dfrac{y}{x+y}$, and find the difference of the results. *Ans.* $\dfrac{x}{x-y}$.

5. Multiply $b + \dfrac{1}{b}$, $b^2 + \dfrac{1}{b^2}$, $b - \dfrac{1}{b}$ together.

The product $= b^4 - \dfrac{1}{b^4}$.

6. $\left(\dfrac{x}{a} + \dfrac{a}{y}\right) \div \left(\dfrac{y}{a} + \dfrac{a}{x}\right) = \dfrac{x}{y}$.

7. $\left(x + \dfrac{xy}{x+y}\right)\left(x - \dfrac{xy}{x+y}\right)\dfrac{x^2-y^2}{x^2+y^2}$

$$= x^3\frac{x^2+xy-2y^2}{(x+y)(x^2+y^2)}.$$

8. $\left(\dfrac{x^3}{y} - \dfrac{y^3}{x}\right) \div \left(\dfrac{x}{y} - \dfrac{y}{x}\right) = x^2+y^2$.

9. Divide $\frac{x^3}{y^3} - \frac{y^3}{x^3}$ by $\frac{x}{y} - \frac{y}{x}$. *Ans.* $\frac{x^2}{y^2} + 1 + \frac{y^2}{x^2}$.

10. Divide $\frac{a^3}{b^3} + \frac{b^3}{a^3}$ by $\frac{a}{b} + \frac{b}{a}$. *Ans.* $\frac{a^2}{b^2} - 1 + \frac{b^2}{a^2}$.

11. Divide $\frac{x}{a} - 1 - \frac{b}{a} - \frac{b^2}{a^2} + \frac{b}{x} + \frac{b^2}{x^2}$ by $x - a$.

The result $= \frac{ax^2 - bax - b^2x - ab^2}{a^2x^2}$.

12. The product of two algebraical expressions is

$$\frac{x^2}{y^2} - \frac{2x}{y} - 3 + 4\frac{y}{x} + 2\frac{y^2}{x^2}$$

and one of them is $\frac{x}{y} - \frac{2y}{x}$. Find the other.

Ans. $\frac{x}{y} - 2 - \frac{y}{x}$.

119. As was observed in (80), every example of the multiplication of fractions supplies one or more examples of division, and *vice versâ*.

INVOLUTION AND EVOLUTION OF FRACTIONS.

120. Since involution, as far as it is at present viewed, is nothing but a repetition of multiplication,

$$\left(\frac{a}{b}\right)^2 = \frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2},$$

$$\left(\frac{a}{b}\right)^3 = \left(\frac{a}{b}\right)^2 \times \frac{a}{b} = \frac{a^2}{b^2} \cdot \frac{a}{b} = \frac{a^3}{b^3},$$

and generally when n is any positive integer,

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

121. Evolution again, being the production of a quantity which when raised to a certain power will produce a given fraction, is performed by extracting the required root of the numerator and denominator separately.

Since $\frac{\sqrt{a}}{\sqrt{b}}$ squared or $\frac{\sqrt{a}}{\sqrt{b}} \cdot \frac{\sqrt{a}}{\sqrt{b}}$ makes $\frac{a}{b}$ (89),

$\therefore \quad \frac{\sqrt{a}}{\sqrt{b}}$ is the square root of $\frac{a}{b}$ or $=\sqrt{\frac{a}{b}}$.

So $\quad \frac{\sqrt[3]{a}}{\sqrt[3]{b}} \cdot \frac{\sqrt[3]{a}}{\sqrt[3]{b}} \cdot \frac{\sqrt[3]{a}}{\sqrt[3]{b}} = \frac{a}{b}$,

$\therefore \quad \frac{\sqrt[3]{a}}{\sqrt[3]{b}} = \sqrt[3]{\frac{a}{b}}$.

And this reasoning may be extended to any root whose exponent is a positive integer.

122. *Examples for Practice.*

Involution.

1. $\left(x+\frac{1}{x}\right)^2 = x^2+\frac{1}{x^2}+2.$

2. $\left(x-\frac{1}{x}\right)^3 = x^3-3x+\frac{3}{x}-\frac{1}{x^3}.$

3. $\left(\frac{a}{b}+\frac{b}{a}-\frac{1}{2}\right)^2 = \frac{a^2}{b^2}+\frac{b^2}{a^2}-\frac{a}{b}-\frac{b}{a}+\frac{9}{4}.$

Evolution.

1. $\sqrt{\frac{9a^4}{16b^2}} = \pm\frac{3a^2}{4b}$ (93).

2. $\sqrt[3]{\frac{27x^3}{3y^4}} = \frac{3x}{y\sqrt[3]{3y}}.$

3. $\sqrt{\frac{a^2+4ab+4b^2}{x^4}} = \pm\frac{a+2b}{x^2}.$

4. $\sqrt{\dfrac{x^2+6xy+9y^2}{x^2-6xy+9y^2}} = \pm\dfrac{x+3y}{x-3y}.$

123. To extract the square root of

$$(x+y)^2+2\frac{(x+y)^2}{xy}+\left(\frac{1}{x}+\frac{1}{y}\right)^2.$$

The given expression is

$$(x+y)^2+2(x+y)\cdot\frac{x+y}{xy}+\left(\frac{1}{x}+\frac{1}{y}\right)^2$$

$$=(x+y)^2+2(x+y)\left(\frac{1}{x}+\frac{1}{y}\right)+\left(\frac{1}{x}+\frac{1}{y}\right)^2$$

$\therefore$ its square root is

$$\pm\left\{x+y+\frac{1}{x}+\frac{1}{y}\right\}\ (96),$$

which may receive the form

$$\pm\left\{x+y+\frac{x+y}{xy}\right\}$$

$$=\pm(x+y)\left(1+\frac{1}{xy}\right).$$

124. To extract the square root of

$$\frac{x^2}{y^2}+\frac{y^2}{x^2}-\left(\frac{x}{y}+\frac{y}{x}\right)+\frac{9}{4}.$$

This expression admits the form :

$$\frac{x^2}{y^2}+2+\frac{y^2}{x^2}-\left(\frac{x}{y}+\frac{y}{x}\right)+\frac{1}{4},$$

$$=\frac{x^2}{y^2}+2\frac{x}{y}\cdot\frac{y}{x}+\frac{y^2}{x^2}-\left(\frac{x}{y}+\frac{y}{x}\right)+\frac{1}{4},$$

$$=\left(\frac{x}{y}+\frac{y}{x}\right)^2-2\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right)+\left(\frac{1}{2}\right)^2,$$

$$=\left(\frac{x}{y}+\frac{y}{x}-\frac{1}{2}\right)^2\ (96).$$

Hence its square root is $\pm\left(\dfrac{x}{y}+\dfrac{y}{x}-\dfrac{1}{2}\right)$.

125. If $\frac{A}{x}=\frac{B}{y}=\frac{C}{z}$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$,

prove that $\frac{A^2}{a^2}+\frac{B^2}{b^2}+\frac{C^2}{c^2}=\frac{A^2+B^2+C^2}{x^2+y^2+z^2}$.

Science Examination 1864.

Let each of the three equal quantities

$\frac{A}{x}$, $\frac{B}{y}$, $\frac{C}{z}$, be supposed $= k$.

$$\text{Then } \left.\begin{aligned} A &= kx \\ B &= ky \\ C &= kz \end{aligned}\right\}.$$

$$\therefore\ A^2+B^2+C^2 = k^2(x^2+y^2+z^2).$$

$$\therefore\ k^2=\frac{A^2+B^2+C^2}{x^2+y^2+z^2}.$$

Again,

$$\frac{A}{a}=k\frac{x}{a} \qquad \therefore\ \frac{A^2}{a^2}=k^2\frac{x^2}{a^2}.$$

$$\frac{B}{b}=k\frac{y}{b} \qquad \therefore\ \frac{B^2}{b^2}=k^2\frac{y^2}{b^2}.$$

$$\frac{C}{c}=k\frac{z}{c} \qquad \therefore\ \frac{C^2}{c^2}=k^2\frac{z^2}{c^2}.$$

$$\therefore\ \frac{A^2}{a^2}+\frac{B^2}{b^2}+\frac{C^2}{c^2}=k^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)$$

$$=\frac{A^2+B^2+C^2}{x^2+y^2+z^2}.$$

CHAPTER III.

SIMPLE EQUATIONS.

126. The letters or symbols in algebra can represent for us either known or unknown quantities; that is, either quantities already ascertained in value, *given* quantities as they are termed, or else quantities which wait to be ascertained and calculated. If it were asked, for instance, what is the length of a string which when an eighth part has been cut off becomes 49 inches, in this case, the final length of the string, 49 inches, is a known or given quantity, while its original length is as yet an unknown or required quantity until some process of calculation shows this to be 56 inches.

Now the great help which algebra offers in calculations is in enabling us to reason upon unknown quantities as if they were known, and to trace the result of certain conditions upon them, while their values are yet waiting to be determined. This will be understood as the present chapter proceeds.

127. *Obs.*—It is customary to represent known quantities by earlier letters of the alphabet, a, b, c, &c., and unknown quantities by the concluding letters of the alphabet, u, v, x, y, z.

128. Suppose the following question presented: There is a string from which if 4 feet be first cut off, and then a third of the remainder be cut off, the length finally left is 10 feet, what is the original length of the string? Let the original length of the string be represented by x feet, a quantity open at present to signify any length whatever. On this unknown quantity we shall trace the result of the curtailments which the string is said in the question to undergo. First, 4 feet are taken from it. The length left is

$x-4$ feet. Now a third part of this remainder is cut off, and two-thirds consequently left, expressed by $\frac{2}{3}(x-4)$ feet. This final remainder, we are informed, is 10 feet, and hereby we have the means of making x, which is thus far open and general, take the particular value of the length of the string with which we are concerned. For it appears that

$$\begin{array}{rlr} & \text{two-thirds of } (x-4) \text{ is to mean} & 10 \text{ feet.} \\ \therefore & \text{one-third of } (x-4) \text{ must mean} & 5 \text{ feet.} \\ \therefore & x-4 \quad\quad ,, & 15 \text{ feet.} \\ \therefore & x \quad\quad\quad ,, & 19 \text{ feet.} \end{array}$$

Thus 19 feet is pronounced to be the original length of the string.

Suppose the question had thus been proposed: There is a string from which if a feet be first cut off, and then a third of the remainder be cut off, the length finally left is b feet, a and b meaning some lengths known but not assigned in figures. What is the original length of the string? If, as before, the original length be termed x feet, the string becomes, after the first curtailment, $x-a$ feet long. Of this a third is taken off and two-thirds left, or $\frac{2}{3}(x-a)$. Now x, which is as yet open to take any value, will be fixed to the particular value of the length of this string of the problem by the condition that

$$\begin{array}{rlccl} & \frac{2}{3}(x-a) \text{ feet means } b \text{ feet.} & & & \\ \therefore & \frac{1}{3}(x-a) & ,, & ,, & \frac{b}{2} \text{ feet.} \\ \therefore & x-a & ,, & ,, & \frac{3b}{2} \text{ feet.} \\ & x & ,, & ,, & \frac{3b}{2}+a \text{ feet.} \end{array}$$

It will be seen that if $a=4$, $b=10$, this reasoning step by step includes that of the former question, which is a particular case of this more general question. By giving any values we please to a and b, other particular problems of the same type result.

129. Now a prominent purpose of algebra is to reduce a problem respecting magnitudes to an equation or equations, an equation being an expression of the equality of two algebraical quantities. Hereby the unknown quantities are connected with those which are known, and by this connection become capable of determination. When the unknown quantities of an equation or system of equations are by proper processes separately expressed in terms of known quantities, the equation or equations are said to be *solved.* If numerical values be the known quantities, as in the former of the two instances just given, the unknown quantities are expressed in numerical form also.

130. *Def.*—In an equation or expression of equality, the two parts declared to be equal, and connected by the sign ($=$) of equality, are called *members* of the equation. Thus, if $x^2-3x=ax-c$ be an equation, x^2-3x is called its former member, and $ax-c$ its latter member.

131. *Obs.*—An equation wherein x is the unknown quantity to be ascertained from it, is sometimes, for brevity, called 'an equation in x.'

132. *Def.*—The exhibiting the value of the unknown quantity in an equation in terms of the known quantities (126) is called solving the equation. The value of the unknown quantity thus obtained is called a solution or a root of the equation.

Thus $x=3$ is a root or solution of the equation

$$\frac{x^2-8}{4}+\frac{x}{2}=\frac{x-3}{7}+\frac{x+11}{4},$$

because if this value 3 is substituted for x

$$\frac{x^2-8}{4}+\frac{x}{2}=\frac{9-8}{4}+\frac{3}{2}=\frac{1}{4}+\frac{3}{2}=\frac{7}{2},$$

and $$\frac{x-3}{7}+\frac{x+11}{4}=\frac{3-3}{7}+\frac{3+11}{4}=\frac{14}{4}=\frac{7}{2},$$

and the two sides or members of the equation are shown to be equal.

133. A root or solution of an equation is said to 'satisfy' the equation.

In the preceding instance if $x=2$ be tried, it will appear that this value is not a root or does not satisfy the equation. For in this case

$$\frac{x^2-8}{4}+\frac{x}{2}=\frac{4-8}{4}+\frac{2}{2}=-1+1=0,$$

$$\frac{x-3}{7}+\frac{x+11}{4}=\frac{2-3}{7}+\frac{2+11}{4}=-\frac{1}{7}+\frac{13}{4},$$

so that the two members of the equation are not made equal by this value 2 being given to x.

Ex. 1. Is any of the values 1, 2, or 3 a root of the equation

$$x^3-2x^2+x-2=0?$$

Ex. 2. Does $x=7$ satisfy the equation

$$\frac{3x-4}{5x-7}+\frac{18}{x+3}=x^2+4?$$

Ex. 3. Is $x=4$ a root of the equation

$$25(1+x^3)=13(1+x)^3?$$

SOLUTION OF EQUATIONS.

134. Equations are solved by application of the following principles.

If two quantities be equal to one another then also are the quantities equal to one another which result from—

i. Adding the same quantity to each of the original pair;

ii. Subtracting the same quantity from each of the original pair;

iii. Multiplying each of the original pair by the same multiplier;
iv. Dividing each of the original pair by the same divisor;
v. Raising each of the original pair to the same power;
vi. Extracting the same root of each of the original pair.

These are axioms which are to be admitted upon no evidence but that which the nature of quantity and of the operations specified conveys to the mind.

Thus if p and q represent the members (130) of an equation, each meaning an expression which may contain known and unknown quantities in any variety of formation, so that $p=q$ is the equation:

i. states that if k be any quantity,

$$p+k = q+k.$$

ii. " $p-k = q-k.$

iii. " $kp = kq.$

iv. " $\frac{p}{k} = \frac{q}{k}.$

v. " if k be any integer, $p^k=q^k$.

vi. " a value of $\sqrt[k]{p}=$ a value of $\sqrt[k]{q}$ (93).

135. *Cancelling.*—From i. and ii. it follows that when any the same quantity appears with the same sign in both members of an equation, it may be cancelled in both, and the remaining quantities are equal still.

Ex. If $x^3+2x^2+3 = 2x^2+4,$

$\therefore$ cancelling $2x^2+3$ from each member we have

$$x^3 = 1.$$

If $x^2-3x+2 = x^3-3x,$

then $x^2+2 = x^3.$

136. *Transposition.*—By i. and ii. a quantity may be transposed from one side of an equation to the other, if only its sign be changed.

$$\text{If} \quad x^3-2x^2+3 = x^4+x,$$

add to each member $2x^2$ (i)

$$\text{then} \quad x^3-2x^2+3+2x^2 = x^4+x+2x^2,$$

$$\text{or} \quad x^3+3 = x^4+x+2x^2;$$

so that the term $2x^2$, which stood with a negative sign in the former member, has been made to stand with a positive sign in the latter member (130).

$$\text{Or if} \quad x^4+2x+3 = x^2,$$

subtract $2x$ from each side (ii.):

$$\text{then} \quad x^4+2x+3-2x = x^2-2x,$$

$$\text{or} \quad x^4+3 = x^2-2x,$$

so that the term $2x$ has been transposed from one side of the equation to the other with its sign reversed.

137. In virtue of iii. the signs of each member of an equation can be simultaneously changed. For if $p=q$ represents the equation, let each side be multiplied by -1, and $-p = -q$ is the resulting equation.

Ex. $$\text{If} \quad x^4-2x^2+1 = x^3-3x,$$

$$\text{then} \quad -x^4+2x^2-1 = -x^3+3x.$$

138. By application of iii. an equation whose members contain fractions may be relieved from the fractional form.

If a single fraction appears, let each member be multiplied by its denominator.

Thus, if $$\frac{x+3}{4}+x = x^2+3,$$

then when both members are multiplied by 4,

$$x+3+4x = 4x^2+12,$$

and no fraction appears.

If there be several fractions, let each member be multiplied by a multiplier which includes all their denominators.

$$\text{If} \quad \frac{x+3}{4}-\frac{x-2}{3}=\frac{x+1}{8},$$

let each member be multiplied by 24, which includes 4, 3, and 8, then

$$6x+18-8x+16=3x+3 \ (42).$$

The smallest number including 4, 3, and 8 has been used as a multiplier. Any larger number including them, as 48 or 72, would have had the same effect of clearing the equation of fractions, but would have been less convenient because it would have introduced larger numbers.

139. *Caution.*—The members of an equation after being altered in any of the ways specified above, do not continue equal to their original values, though they continue to be equal to one another. For instance, if

$$x^2+4=x^3,$$

also it is true that

$$x^2+4x+4=x^3+4x,$$

but it is not necessarily true, or intended to be implied, that either x^2+4 is equal to x^2+4x+4 or x^3 is equal to x^3+4x.

140. By the use of these principles an equation has to be so transformed, perhaps by many successive alterations, that eventually the first member is the unknown quantity and the second member some known quantities, which therefore the unknown quantity is shown to have for its value. It is impossible to give rules beyond this general description of the method of solving equations. Practice and observation of examples suggest the methods to be used in any particular case for disengaging the unknown quantity and exhibiting its value; on which account several examples fully worked out will be presently given.

SIMPLE EQUATIONS.

141. *Def.*—When an equation, cleared, if necessary, of fractions or roots affecting the unknown quantity, contains the first power only of that unknown quantity, it is called a simple equation or an equation of the first degree.

Thus $3x+7=5x+3$ is a simple equation, the first power only of x appearing, but $2x^2+x=1$ is not a simple equation, because it contains the second as well as the first power of x.

142. A simple equation has but one root. For the form to which it can be reduced is $ax = b$, a and b being certain known quantities, whence $x = \frac{b}{a}$, and to suppose x admitting more than one value would require that $\frac{b}{a}$, a known and determined quantity, should have more values than one, which it cannot have. In advancing with the subject of Algebra the student will arrive at equations which have more than one root. For instance, he will find by substitution that the equation $x^3-6x^2+11x=6$, is satisfied if x is either 1, 2, or 3.

143. *Obs.*—The term *simple* is not used with any reference to the easiness or difficulty of solving equations, as if simple equations were necessarily easier to solve than those of higher degrees, but in the sense in which *simple* is contrasted with double, triple, multiple.

144. Some examples shall now be given of separating the unknown quantity in a simple equation from other quantities with which it appears in combination, and thus exhibiting its value. They are no more than examples in particular instances, and the student will have to apply and combine such processes as are here used, accordingly as in any equation before him they seem to promise success.

145. Let the equation be proposed

$$3x-4 = 2x+1.$$

It will be observed that in containing the first power only of x, this equation is by the definition a simple equation (141).

To solve this equation, the object is to bring x equal to some numerical quantity (140). We have a motive therefore for collecting on one side of the equation every term which involves x. This will be effected by the principle of transposition (136). First by transposition of $2x$:

$$3x-4-2x = 1.$$

Next to remove from the side containing x every numerical quantity connected with it by addition or subtraction, by transposition of the term -4 we have,

$$3x-2x = 1+4,$$

$$\text{or} \quad x = 5.$$

Thus 5 is the solution or root of this equation, and the truth of this result can be tested by substituting the value 5 for x in the equation as it originally appears. Then its members become

$$3x-4 = 15-4 = 11,$$

$$2x+1 = 10+1 = 11,$$

and their equality is proved, or the equation is satisfied (133) by $x = 5$.

The practical usefulness of an equation such as this which has been solved may be better valued if it is observed that this equation leads to the answer of the following among other questions.

What is the number which, whether we take 4 from its triple or add 1 to its double, gives the same result?

If x represents the number, as yet unknown, it will be observed that the equation just solved states the fact which defines it.

146. $$5(4-x)+3(x-6) = 10.$$

Science Examination 1868.

When the operations indicated by the brackets are effected,

$$20-5x+3x-18 = 10.$$

When the terms containing x are made to stand in one member of the equation, and the rest of the terms transposed into the other member,

$$\begin{aligned} -5x+3x &= 10+18-20, \\ \text{or}\quad -2x &= 8, \\ -x &= 4 \text{ (134, iv.)} \\ x &= -4 \text{ (137)}. \end{aligned}$$

147. $$x-5-4\left(\frac{3a}{2}-4\right) = 3\,(15-2a).$$

With the view of placing x on one side of the equation and known quantities on the other, we transpose from the first member all that is known.

$$\begin{aligned} \therefore\ x &= 3(15-2a)+5+4\left(\frac{3a}{2}-4\right) \\ &= 45-6a+5+6a-16 \\ &= 34. \end{aligned}$$

148. The principles thus far exemplified will suffice for solving the following equations :

1. $x+3 = 4(x-9)$. *Ans.* $x = 13$.
2. $5(x+7) = 8(x+1)$. *Ans.* $x = 9$.
3. $3(x-1) = 4(x+3)+3$. *Ans.* $x = -18$.
4. $12(x-3) = 10(x+3)$. *Ans.* $x = 33$.
5. $16-x-[7x-\{8x-(9x-\overline{3x-6x})\}] = 0$. *Ans.* $x = \frac{4}{3}$.
6. $a(ax+b)-b(bx+a) = c^2$. *Ans.* $x = \dfrac{c^2}{a^2-b^2}$.
7. $x-a-3(x-b) = 2a$. *Ans.* $x = \frac{3}{2}(b-a)$.
8. $a(x-a^2)+b(x-b^2) = 0$. *Ans.* $x = a^2-ab+b^2$.
9. $x-a-\{x-(2x-b)\} = a-b$. *Ans.* $x = a$.
10. $m^2-n-mx = n^2-m-nx$. *Ans.* $x = m+n+1$.

149. Let the equation to be solved be

$$\frac{x-4}{3}-\frac{7-x}{4}=\frac{5}{4}.$$

Since the result to be aimed at is the expressing x as equal to some numerical quantity, it will conduce to this purpose if the equation is first cleared of fractions (138) by being multiplied throughout by 12.

$$\therefore\ 4x-16-21+3x=15\ (42).$$

To have the terms which contain x standing alone on one side of the equation, it is convenient to transpose the terms -16 and -21 (136).

$$\therefore\ 4x+3x=15+16+21,$$

$$\text{or}\qquad 7x=52.$$

Then if each side of the equation be divided by 7 (134, iv.)

$$x=\frac{52}{7}=7\tfrac{3}{7}.$$

If this value is substituted for x in the former member of the equation, this member becomes

$$\begin{aligned}\frac{7\frac{3}{7}-4}{3}-\frac{7-7\frac{3}{7}}{4}&=\frac{3\frac{3}{7}}{3}+\frac{1}{4}\cdot\frac{3}{7}\\&=1\tfrac{1}{7}+\tfrac{3}{28}\\&=1+\tfrac{7}{28}\\&=1+\tfrac{1}{4}=\tfrac{5}{4},\end{aligned}$$

or the equation is satisfied.

150. $$\frac{x+2}{3}+\frac{x+3}{2}=x.$$

Science Examination 1867.

To relieve this equation of fractions let its members be multiplied by 6, a quantity chosen as the least which will include the denominators 3 and 2 (138).

$$\therefore\ 2x+4+3x+9=6x,$$

$$\text{or}\qquad 5x+13=6x.$$

If $5x$ be now transposed for the purposes of collecting in one member all the terms which contain x,

$$x = 13.$$

151. $\dfrac{x}{10} + 10x = \dfrac{x}{2} + \dfrac{x}{5} + \dfrac{x}{40} - \dfrac{10-x}{7} + 93\frac{3}{4}.$

Science Examination 1866.

To remove fractions from the equation we multiply by a number which will include all the denominators. Such a number, their least common multiple, is 280. Any other common multiple of the denominators would answer the purpose, but would be less convenient than 280 by reason of its introducing larger numbers.

Then $28x + 2800x = 140x + 56x + 7x - 400 + 40x + 26250.$
By transposition

$$28x + 2800x - 140x - 56x - 7x - 40x = 26250 - 400,$$
$$2585x = 25850,$$
$$x = 10.$$

152. $\dfrac{x-5a}{4a} - \dfrac{x-3a}{9} = \dfrac{a}{18}.$

With the purpose of removing the denominators of the fractions, let each member of the equation be multiplied by $36a$, which includes all the three denominators.

$$\therefore\ 9x - 45a - 4ax + 12a^2 = 2a^2.$$

Transpose from the left hand member the two of its terms in which x does not appear (136).

$$\therefore\ 9x - 4ax = 2a^2 + 45a - 12a^2,$$

which is the same as

$$(9-4a)x = 45a - 10a^2.$$

Here then x is obtained, only that it is multiplied by a coefficient. If then both members be divided by this coefficient (134, iv.)

$$x = \frac{45a - 10a^2}{9-4a}.$$

153. By these models the following may be solved:

1. $\frac{7-x}{2}-\frac{9-2x}{3}=1.$ *Ans.* $x=3.$

2. $\frac{x+1}{4}-\frac{x-2}{5}=\frac{2}{3}.$ *Ans.* $x=\frac{1}{3}.$

3. $\frac{1-2x}{15}-\frac{1-x}{16}=\frac{1}{2}.$ *Ans.* $x=-7.$

4. $\frac{5x-4}{9}-\frac{2x-13}{5}=\frac{x+1}{3}.$ *Ans.* $x=10\frac{1}{4}.$

5. $\frac{4x+1}{5}-\frac{3x-5}{7}=\frac{x+4}{3}.$ *Ans.* $x=11.$

6. $\frac{x+5}{4}+\frac{3x+4}{5}=\frac{7x-1}{6}.$ *Ans.* $x=7.$

7. $\frac{4x+7}{6}-\frac{5x-54}{4}=\frac{1}{12}.$ *Ans.* $x=25.$

8. $\frac{2x-5}{3}+x=\frac{3x-2}{5}+3.$ *Ans.* $x=4.$

9. $\frac{2x-6}{5}-\frac{x-4}{9}-\frac{3x}{13}=0.$ *Ans.* $x=13.$

10. $x-\frac{x-2}{3}=\frac{x+23}{4}-\frac{10+x}{5}.$ *Ans.* $x=5.$

11. $x+\frac{x-7}{3}-\frac{3x+4}{5}+\frac{2x}{7}=4.$ *Ans.* $x=7.$

12. $x+\frac{x+4}{2}-\frac{3x-4}{5}-\frac{x}{8}=9.$ *Ans.* $x=8.$

13. $\frac{x+3}{2}-\frac{3x-1}{20}-3\frac{1}{5}=\frac{11-x}{5}.$ *Ans.* $x=7.$

14. $\frac{x-1}{7}+\frac{23-x}{5}=7-\frac{4+x}{4}.$ *Ans.* $x=8.$

15. $\frac{17-\frac{3}{2}x}{5}-\frac{x+1}{1\frac{1}{2}}=\frac{29-\frac{11}{2}x}{3}.$ *Ans.* $x=8.$

16. $\frac{5x-1}{2}-\frac{7x-2}{10}=6\frac{3}{5}-\frac{x}{2}.$ *Ans.* $x=3.$

17. $\frac{x}{2}+\frac{x}{\ }-\frac{x}{4}=\frac{7}{10}$. *Ans.* $x=\frac{6}{5}$.

18. $\frac{x-1}{3}+\frac{4x-\frac{3}{4}}{5}-\frac{7x-6}{8}-\frac{3x-9}{10}=2+\frac{x-2}{2}$.

Ans. $x=\frac{4}{13}$.

19. $\frac{5x}{6}+\frac{1}{4}-\frac{x}{3}=x-3$. *Ans.* $x=12$.

20. $\frac{1}{6}\left(x+3\right)-\frac{1}{7}\left(11-x\right)=\frac{2}{5}\left(x-4\right)-\frac{1}{21}\left(x-3\right)$.

Ans. $x=9$.

21. $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}+\frac{x}{5}=7\frac{5}{6}$. *Ans.* $x=10$.

22. $\frac{5x-3}{7}-\frac{8-x}{3}=\frac{7}{2}x-\frac{4}{5}\left(4x+2\right)$. *Ans.* $x=2$.

23. $\frac{7x+5}{23}+\frac{9x-1}{10}-\frac{x-9}{5}+\frac{2x-3}{15}=23\frac{1}{2}$.

Ans. $x=\frac{3006}{157}$.

24. $\frac{3\frac{1}{4}-4x}{1\frac{1}{3}}-\frac{49}{54}\left(3\frac{1}{2}-5x\right)=\frac{7}{16}+\frac{55}{108}(3x-2)$.

Ans. $x=17$.

25. $\frac{6x-1}{15}-\frac{9x-2}{16}=\frac{x-4}{6}-\frac{x+4}{8}$. *Ans.* $x=6$.

26. $\frac{x+9}{3}-\frac{x-5}{4}-\frac{x}{4\frac{1}{2}}=2+\frac{2x-7}{11}$. *Ans.* $x=\frac{1143}{127}$.

27. $\frac{x}{8}-\frac{x-1}{2\frac{1}{2}}=\frac{3x-4}{15}+\frac{x}{12}$. *Ans.* $x=\frac{80}{67}$.

28. $\frac{x-a}{3}-\frac{x-b}{5}=3b+a$. *Ans.* $x=10a+21b$.

29. $\frac{x-m^2}{4}+\frac{x-n^2}{6}=\frac{7}{12}mn$. *Ans.* $x=\frac{(3m+n)(m+2n)}{5}$.

154. $(x-2)(x-3)+(x-1)(x-3)+(x-1)(x-2)$
$$=3x^2-1.$$

When the multiplications expressed by the bracketed quantities are effected, the equation stands,

$$x^2-5x+6+x^2-4x+3+x^2-3x+2=3x^2-1.$$

At present this appears not to fulfil the character of a simple equation, since x^2 as well as x appears in it, but when the terms are collected,

$$3x^2-12x+11=3x^2-1,$$

and when the term $3x^2$ common to both sides is cancelled,

$$-12x+11=-1,$$

so that the equation proves to be a simple equation.

To have the term with x standing by itself we transpose 11, and

$$-12x=-1-11=-12.$$

Then divide each side by -12 and

$$x=1.$$

155. The following is another instance of an equation not appearing at first to be a simple equation, but proving to be reducible to one.

$$\frac{5x^2+x-3}{5x-4}=\frac{7x^2-3x-9}{7x-10},$$
$$(5x^2+x-3)(7x-10)=(7x^2-3x-9)(5x-4),$$
$$35x^3-43x^2-31x+30=35x^3-43x^2-33x+36,$$
$$33x-31x=36-30,$$
$$2x=6,$$
$$x=3.$$

156. $(3x-1)^2+(4x-2)^2=(5x-3)^2.$

Science Examination, 1867.

When the operations indicated are effected

$$9x^2-6x+1+16x^2-16x+4=25x^2-30x+9.$$

Now $9x^2+16x^2$ in the former member, making $25x^2$, is cancelled by the similar term in the second member, and then transposition gives

$$30x-6x-16x=9-1-4,$$
$$8x=4,$$
$$x=\tfrac{1}{2}.$$

157. $(x+3)^2-3x(4x+1) = 5x^2-(4x-5)^2$.

When the operations signified have been effected,

$$x^2+6x+9-12x^2-3x = 5x^2-16x^2+40x-25,$$

$$\text{or} \quad 37x = 34$$

$$x = \tfrac{34}{37}.$$

Science Examination, 1868.

158. $(x+a)(x-b)-(x-a)(x+b) = a^2-b^2$.

When the operations signified are effected,

$$x^2+ax-bx-ab-(x^2-ax+bx-ab) = a^2-b^2,$$

$$\text{or} \quad 2ax-2bx = a^2-b^2,$$

$$\text{or} \quad 2(a-b)x = a^2-b^2;$$

$$\therefore\ x = \frac{a^2-b^2}{2(a-b)},$$

$$= \frac{(a-b)(a+b)}{2(a-b)},$$

$$= \frac{a+b}{2}.$$

159. $$\frac{3x+1}{x+1} = \frac{3bx-2a+c}{b(x+1)-a}.$$

If this equation be cleared of fractions,

$$3bx(x+1)-3ax+b(x+1)-a=3bx(x+1)-(2a-c)(x+1),$$

or when the term $3bx(x+1)$ common to both sides is cancelled,

$$-3ax+bx+b-a = -2ax+cx-2a+c,$$

$$\therefore\ x(a-b+c) = a+b-c,$$

$$x = \frac{a+b-c}{a-b+c}.$$

160. $$\frac{3}{x-a}+\frac{5}{x+a} = \frac{8}{x}.$$

In order, as a first step, to clear this equation of fractions, let it be multiplied throughout by $(x-a)(x+a)x$.

Then,

$$3(x+a)x+5(x-a)x = 8(x-a)(x+a),$$

or when the multiplications are effected

$$8x^2-2ax = 8x^2-8a^2.$$

Now $8x^2$, common to the two sides, may be cancelled,

$$\therefore\ -2ax = -8a^2,$$

$$x = 4a.$$

161.
$$\frac{x(x+3)}{(x+1)(x+2)} + \frac{4}{3x(x+2)} = 1.$$

If to remove the fractional forms both members be multiplied by $3x(x+1)(x+2)$,

$$3x^2(x+3)+4(x+1) = 3x(x+1)(x+2),$$

or when the expressed multiplications are effected,

$$3x^3+9x^2+4x+1 = 3x^3+9x^2+6x,$$

$$\therefore\ 2x = 1,$$

$$x = \tfrac{1}{2}.$$

162. Instead of at once clearing an equation of fractions by multiplying by some quantity which will absorb all the denominators, it is in certain cases expedient to unite two or more terms as a preliminary step. Practice only will show when this method is serviceable.

Ex.
$$\frac{4x+2}{9} = \frac{8x+19}{18} - \frac{7x-29}{5x-12}.$$

By transposition

$$\frac{7x-29}{5x-12} = \frac{8x+19}{18} - \frac{4x+2}{9}$$

$$= \frac{8x+19}{18} - \frac{8x+4}{18} = \frac{15}{18} = \frac{5}{6}.$$

$$\therefore\ 42x-174 = 25x-60,$$

$$17x = 174-60 = 114,$$

$$x = \tfrac{114}{17} = 6\tfrac{12}{17}.$$

163. The following example will show the expediency of combining the terms of an equation in such order as to reduce the labour of the process.

$$\frac{x^2+2x+2}{x+1}+\frac{x^2+8x+20}{x+4}=\frac{x^2+4x+6}{x+2}+\frac{x^2+6x+12}{x+3},$$

$$\frac{x^2+8x+20}{x+4}-\frac{x^2+6x+12}{x+3}=\frac{x^2+4x+6}{x+2}-\frac{x^2+2x+2}{x+1},$$

$$1+\frac{x}{(x+4)(x+3)}=1+\frac{x}{(x+2)(x+1)},$$

$$\therefore\ x(x+2)(x+1)=x(x+4)(x+3).$$

Hence either $x=0$ (133);

or if x has any other value,

$$(x+2)(x+1)=(x+4)(x+3)$$
$$x^2+3x+2=x^2+7x+12,$$
$$4x=10,$$
$$x=\tfrac{5}{2}.$$

164. *Examples for Practice.*

1. $(x+1)(x-2)=(x-3)(x+4)$. *Ans.* $x=5$.
2. $(x+1)(x+2)-(x-1)(x-2)=18$. *Ans.* $x=3$.
3. $(a+x)(b+x)=(c+x)(d+x)$. *Ans.* $x=\dfrac{cd-ab}{a+b-c-d}$.
4. $(x-\overline{a+b})^2=x^2-(a-b)^2$. *Ans.* $x=\dfrac{a^2+b^2}{a+b}$.
5. $\dfrac{x-3}{x-2}=\dfrac{x-2}{x-3}$. *Ans.* $x=\tfrac{5}{2}$.
6. $(x-2a)^2-(x-a)^2=6a^2$. *Ans.* $x=-\dfrac{3a}{2}$.
7. $\dfrac{x-a}{x-b}=\dfrac{x-a}{x-d}$. *Ans.* $x=\dfrac{bc-ad}{b+c-a-d}$.
8. $\dfrac{(x+1)(2x+2)}{x-3)(x+6)}=2$. *Ans.* $x=19$.

9. $\dfrac{\frac{1}{x}-3}{\frac{1}{x}-2}=\frac{1}{2}$. *Ans.* $x=\frac{1}{4}$.

10. $\dfrac{2}{1-5x}=\dfrac{5}{1-2x}$. *Ans.* $x=\frac{1}{7}$.

11. $\dfrac{1}{x}+\dfrac{1}{2x}-\dfrac{1}{3x}=\dfrac{7}{3}$ *Ans.* $x=\frac{1}{2}$.

12. $\dfrac{2x+6}{x+1}+\dfrac{3x-7}{x-2}=5$. *Ans.* $x=3$.

13. $\dfrac{x}{3-x}-\dfrac{18-x}{x}=\dfrac{3}{x}$. *Ans.* $x=\frac{21}{8}$.

14. $2x-\dfrac{3x+5}{x-5}=\dfrac{6x-13}{3}$ *Ans.* $x=20$.

15. $\dfrac{x+1}{7}+x(x+2)=(x+1)^2$. *Ans.* $x=6$.

16. $\dfrac{x+4a+b}{x+a+b}+\dfrac{4x+a+2b}{x+a-b}=5$. *Ans.* $x=-b$.

17. $\dfrac{35}{1-3x}=\dfrac{28}{1-4x}$. *Ans.* $x=\frac{1}{8}$.

18. $\dfrac{2x-1}{7}-\dfrac{2x-3}{5}=7-\dfrac{x+2}{2}$. *Ans.* $x=\frac{388}{27}$.

19. $\dfrac{3x+2}{x-1}+\dfrac{2x-4}{x+2}=5$. *Ans.* $x=6$.

20. $\dfrac{1}{1-x}+\dfrac{2}{2-x}-\dfrac{3}{3-x}=0$. *Ans.* $x=\frac{3}{2}$.

21. $\dfrac{x+5}{7}-\dfrac{x}{2}=\dfrac{4}{35x}-\dfrac{5x}{14}$. *Ans.* $x=\frac{4}{25}$.

22. $\dfrac{(2x+3)x}{2x+1}+\dfrac{1}{3x}=x+1$. *Ans.* $x=1$.

23. $(2x+1)(x+1)=(3x-1)(2x-1)$ *Ans.* $x=0$ or 2.

24. What value of x will make the excess of $(2x+4)(3x+4)$ over $(3x-2)(2x-8)$ equal to 96? *Ans.* $x=2$.

25. If $\frac{2a+n}{3n+69a}$ is $\frac{1}{33}$ when a is $\frac{1}{3}$; what is n?

Ans. n

27. $\frac{(x-a)(x-b)}{(x-c)(x-d)} = \frac{x-a-b}{x-c-d}$.

Ans. $x = \frac{ab(c+d)-cd(a-}{ab-cd}$

165. Let the proposed equation be

$$x = a - \sqrt{2ax+x^2}.$$

If by transposition the term under the root is plac one side of the equation alone, then by squaring both the root, which it is our first object to remove, wil appear.

$$\sqrt{2ax+x^2} = a-x,$$

$$\text{whence} \quad 2ax+x^2 = (a-x)^2,$$

$$= a^2-2ax+x^2.$$

$$\therefore \; 2ax = a^2-2ax,$$

$$4ax = a^2,$$

$$x = \frac{a^2}{4a} = \frac{a}{4}.$$

166. $\sqrt{x+\sqrt{x^2+4}} = 2.$

The unknown quantity is here buried under two sig evolution, which must be removed before it can be mined.

If each side be squared,

$$x+\sqrt{x^2+4} = 4.$$

In order to clear off the sign of evolution still remai we must transpose, so that

$$\sqrt{x^2+4} = 4-x.$$

Then when both sides are again squared,

$$x^2+4 = 16-8x+x^2,$$

or when x^2 is cancelled,

$$4 = 16-8x,$$

$$8x = 16-4 = 12,$$

$$x = \tfrac{12}{8} = \tfrac{3}{2}.$$

167. $\sqrt{8+x}+\sqrt{x+3}=5$.

Before x can be determined the roots under which it appears at present must be removed, and this must be by the process of squaring both sides of the equation (134 v.). In this example, however, it will be found that the process has to be employed twice.

By transposition

$$\sqrt{8+x}=5-\sqrt{x+3}.$$

If each side of the equation be now squared,

$$8+x=28+x-10\sqrt{x+3};$$

$\therefore$ by cancelling x and transposing,

$$10\sqrt{x+3}=28-8=20.$$

By division of each side by 10,

$$\sqrt{x+3}=2.$$

If both sides of the equation be now squared,

$$x+3=4,$$
$$x=1.$$

168. $\sqrt{3x+1}+\sqrt{3x}=4\{\sqrt{3x+1}-\sqrt{3x}\}$.

When the latter member is relieved from the bracket,

$$\sqrt{3x+1}+\sqrt{3x}=4\sqrt{3x+1}-4\sqrt{3x}.$$

The terms are now to be so transposed that those which can be added together shall stand together on the same side;

then $\sqrt{3x}+4\sqrt{3x}=4\sqrt{3x+1}-\sqrt{3x+1}$,

or $5\sqrt{3x}=3\sqrt{3x+1}$.

Now let each side of the equation be squared.

$$\therefore\ 25\times 3x=9(3x+1),$$

or $75x=27x+9$,

and when by transposition the two terms with x are made to stand in the same member,

$$75x - 27x = 9,$$
$$48x = 9,$$
$$\therefore\ x = \tfrac{9}{48} = \tfrac{3}{16}.$$

169. When fractions are equal to one another, then also will the fractions be equal which are formed from each by adding the numerator and denominator to make a new numerator, and taking the difference of the numerator and denominator to make a denominator.

Thus, if $\quad \dfrac{a}{b} = \dfrac{c}{d},$

$$\frac{a}{b} + 1 = \frac{c}{d} + 1,$$

$$\frac{a+b}{b} = \frac{c+d}{d}.$$

Again, $\quad \dfrac{a}{b} - 1 = \dfrac{c}{d} - 1,$

$$\frac{a-b}{b} = \frac{c-d}{d},$$

$$\therefore\ \frac{a+b}{b} \cdot \frac{b}{a-b} = \frac{c+d}{d} \cdot \frac{d}{c-d},$$

or $\quad \dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}.$

This principle will often abridge the work in solution of an equation, as in the following instance :—

$$\frac{ax+a^2-2}{ax-a^2+2} = \frac{b^2+2b+1}{b^2-2b+1},$$

$$\therefore\ \frac{ax+a^2-2+(ax-a^2+2)}{ax+a^2-2-(ax-a^2+2)} = \frac{b^2+2b+1+(b^2-2b+1)}{b^2+2b+1-(b^2-2b+1)}$$

or $\quad \dfrac{2ax}{2a^2-4} = \dfrac{2b^2+2}{4b},$

$$\therefore \quad \frac{ax}{a^2-2} = \frac{b^2+1}{2b},$$

$$ax = \frac{(b^2+1)(a^2-2)}{2b},$$

$$x = \frac{(b^2+1)(a^2-2)}{2ab}.$$

170. *Examples for Practice.*

1. $\sqrt{x+9} = 1+\sqrt{x}.$ *Ans.* $x = 16.$

2. $\sqrt{12+x} = 2+\sqrt{x}.$ *Ans.* $x = 4.$

3. $\sqrt{5+x}+\sqrt{x} = \dfrac{15}{\sqrt{5+x}}.$ *Ans.* $x = 4.$

4. $\sqrt{x}+\sqrt{9+x} = \dfrac{45}{\sqrt{9+x}}.$ *Ans.* $x = 16.$

5. $\dfrac{\sqrt{12+x}+\sqrt{12}}{\sqrt{12+x}-\sqrt{12}} = 2$ (169). *Ans.* $x = 96.$

6. $\dfrac{\sqrt{4x+1}+\sqrt{4x}}{\sqrt{4x+1}-\sqrt{4x}} = 9$ (169). *Ans.* $x = \frac{4}{9}.$

7. $1+2\sqrt{x} = \sqrt{4x+\sqrt{16x+2}}.$ *Ans.* $x = \frac{1}{64}.$

8. $\sqrt{(x+a)^2+2ab+b^2} = b-a-x.$ *Ans.* $x = -2a.$

9. $\sqrt{4+x}-\sqrt{3} = \sqrt{x}.$ *Ans.* $x = \frac{1}{12}.$

10. $\dfrac{5x-9}{\sqrt{5x}+3} = 1+\dfrac{\sqrt{5x}-3}{2}.$ *Ans.* $x = 5.$

11. $\sqrt{x+p}+\sqrt{x+q} = \sqrt{p+q}.$ *Ans.* $x = -\dfrac{pq}{p+q}.$

12. $\sqrt{x+c\sqrt{4x+2c^2}} = c+\sqrt{x}.$ *Ans.* $x = \dfrac{c^2}{16}.$

CHAPTER IV.

PROBLEMS PRODUCING SIMPLE EQUATIONS WITH ONE UNKNOWN QUANTITY.

171. The reader will now see the use of his power of solving equations. Problems relating to number, quantity, and shape, will be made to produce equations, and will be solved by solving these equations. The equations are really verbal statements of facts translated into algebraical language. The power of expressing facts in algebraical terms cannot be given by any rules, but must be gained by practice, and by the study of such examples as will now be presented. The problems which will at present be discussed will require simple equations only for their solution.

172. Of some problems in this chapter the method of solution is given in detail, while the rest are left with answers only for the student's exercise. An attempt has been made to arrange together, as far as possible, questions which appear to embody the same leading idea. The reader is recommended first to peruse the explanations of the solutions as they are given, then, with the book closed, to reproduce on paper these solutions which he has studied, afterwards to proceed to the questions which are left for his exercise.

173. 1. Find a number which when multiplied by 4 will exceed 30 as much as it is now below 30.

Let x represent the number required.

Then $30-x$ is the quantity by which it is below 30 (a). But if the number is multiplied by 4 it becomes $4x$, and $4x-30$ is the quantity by which $4x$ is above 30 (b).

Now the fact known respecting the required number is that the defect (*a*) is equal to the excess (*b*),

$$\therefore\ 30-x = 4x-30.$$

This equation is the expression in algebraical language of the fact which characterises the number required to be found, the fact which defines this number and separates it from other numbers. If the equation be solved by transposition,

$$5x = 60,$$
$$x = 12.$$

This number 12 is under 30 by 18, but if it be multiplied by 4 and becomes 48 it then exceeds 30 by 18.

174. 2. Find a fraction where the denominator exceeds the numerator by 2, while if 3 be added to the denominator the fraction becomes equal to $\frac{1}{2}$.

By the first clause of this statement we may represent the fraction by $\dfrac{x}{x+2}$.

After the alteration stated in the second clause this fraction would become $\dfrac{x}{x+5}$.

Hence by the terms of the question

$$\frac{x}{x+5} = \frac{1}{2},$$
$$\therefore\ 2x = x+5,$$
$$x = 5,$$

and the fraction required is $\frac{5}{7}$.

The addition of 3 to the denominator makes this fraction become $\frac{5}{10}$ which is equal to $\frac{1}{2}$.

175. 3. A person wishes to give some boys 3*d.* each, but has not money enough by 8*d.* If he gives them 2*d.* each he has 3*d.* remaining. How many boys are there?

Let x be the number of boys.

The person who is giving the money to them requires $3x$ pence if he is to give them 3*d.* each, and he has not so

much by 8*d.* Therefore the money which he has is $3x-8$ pence (*a*).

Again, to give the boys 2*d.* each he will spend $2x$ pence, and then he has 3*d.* left; wherefore his money is $2x+3$ pence (*b*).

Thus the conditions of the problem have led us to two expressions, (*a*) and (*b*), for the same sum of money, and thereby supply the equation

$$3x-8 = 2x+3,$$

whence $x = 11$,

or there are 11 boys.

The money which the person has is thence known to be 30 pence or half-a-crown.

176. 4. Divide the number 208 into 2 parts so that the sum of one-fourth of the greater and one-third of the less is less by 4 than 4 times the difference of the parts.

Assume $104+x$ and $104-x$ to be the 2 parts, their sum being 208. Let $104+x$ be the greater.

One fourth of the greater is $\frac{104+x}{4}$,

one third of the less is $\frac{104-x}{3}$,

and the sum of these is $\frac{104+x}{4}+\frac{104-x}{3}$ . . . . (*a*).

Now, the difference of the parts is $2x$, and 4 times this difference is $8x$ . . . . (*b*).

By the question (*a*) is less than (*b*) by 4.

$$\therefore \frac{104+x}{4}+\frac{104-x}{3} = 8x-4.$$

This is the equation expressing the question proposed. If this equation be multiplied throughout by 12 to remove the fractions,

$$312+3x+416-4x = 96x-48,$$
$$97x = 776,$$
$$x = 8,$$

and the parts required are $104+x = 112$,

and $104-x = 96$.

177. 5. A is twice as old as B, and 22 years ago he was 3 times as old. What is A's age?

Let B be x years old, and A accordingly $2x$ years old.

Their ages 22 years ago were $x-22$ and $2x-22$ years respectively.

Hence by the question

$$2x-22 = 3(x-22),$$
$$x = 44.$$

One then is now 44 years and the other 88 years old.

178. 6. A garrison was victualled for 30 days. After 10 days it was reinforced by 3000 men, and then the provisions lasted only 5 days more at the same uniform rate of consumption. What was the original number of the garrison?

Let there be x thousands of men in the garrison at first, and consequently $x+3$ thousands after the reinforcement.

At the end of 10 days the provisions are sufficient to feed x thousands of men during 20 days,

$\therefore$ to feed 1000 of men during $20x$ days,

or „ $x+3$ thousands „ $\dfrac{20x}{x+3}$ „

But the statement before us makes this quantity of provisions suffice for $x+3$ thousands of men during 5 days,

$$\therefore \frac{20x}{x+3} = 5,$$
whence $x = 1$,

or there were 1000 men in the garrison originally.

179. 7. A cistern which can hold 820 gallons is filled in 20 minutes by 3 pipes, which let water into it at uniform rates. The first pipe admits 10 gallons more than the third pipe, and the second pipe admits 5 gallons less than

the third pipe every minute. How much water flows through each pipe in a minute?

Let the third pipe admit x gallons of water into the cistern every minute.

Then by the terms of the question the first pipe admits $x+10$ gallons, and the second admits $x-5$ gallons every minute.

Hence the quantities of water which the first, second, and third pipes admit in 20 minutes are, respectively,

$$\begin{array}{ll} 20(x+10) & \text{gallons,} \\ 20(x-5) & \quad ,, \\ 20x & \quad ,, \end{array}$$

Together therefore they admit in 20 minutes

$$20(x+10)+20(x-5)+20x \text{ gallons.}$$

But in 20 minutes, we are informed, they fill the cistern, i.e. they let in 820 gallons,

$$\therefore\ 20(x+10)+20(x-5)+20x=820.$$

If this equation be divided throughout by 20,

$$x+10+x-5+x = 41,$$
$$\text{whence } x = 12,$$

or the first pipe lets in $x+10 = 22$ gallons every minute,
,, second ,, $x-5 = 7$,, ,,
,, third ,, $x = 12$,, ,,

180. 8. The sum of 800*l.* is to be divided between three persons so that their shares shall be as the numbers 3, 4, and 5. What did each receive?

In conformity with the relation which their shares are to have, we may represent their shares by $3x$, $4x$, and $5x$ pounds.

They receive, then, together $3x+4x+5x$ or $12x$ pounds, and what they receive together must be the sum which is to be divided among them,

$$\therefore\ 12x = 800,$$
$$x = \tfrac{200}{3}l.;$$

$\therefore$ the share of the first $3x = 200l.$,

" second $4x = \frac{800}{3}l. = 266l.\ 13s.\ 4d.$,

" third $5x = \frac{1000}{3}l. = 333l.\ 6s.\ 8d.$

Obs.—Though these questions are selected as easy examples of the use of algebra, they may not all be out of the reach of arithmetic. This problem and the one before it can be solved by arithmetical considerations alone without appeal to algebra.

181. 9. In a basket of apples one in every ten is bad, and the rest sold at the rate of three for twopence give the owner a shilling more than if he had sold the whole at a halfpenny each. Find the number of apples.

Suppose that there are $10x$ apples, so that $9x$ are good.

If 3 are sold for *2d.* or 9 for *6d.* the $9x$ apples are sold for $6x$ pence.

If all were sold at a halfpenny each, they would produce $5x$ pence.

Hence, by the terms of the question,

$$6x = 5x + 12,$$
$$x = 12,$$
$$\therefore\ 10x = 120,$$

or there are 120 apples.

If it be asked why various assumptions are made for the thing required in these problems, the answer is that it would be quite possible to solve them by allowing x in every instance to represent the quantity sought, but in cases where $10x$, $104+x$, or the like have been adopted to represent the result to be determined, it has been for some foreseen convenience of making the numbers introduced smaller, and the numerical processes by consequence less troublesome. Each of these problems may be tried by other assumptions, and the convenience of the assumptions chosen will, it is expected, become apparent.

182. 10. On a certain day 58682 persons entered the Exhibition of 1862, some paying a shilling each for admission and the rest presenting season tickets. The money taken was a number of pounds and seven shillings over, and the number of pounds was less by 295 than the number of persons who entered with season tickets. Find the number of persons who entered respectively by payment and by season tickets.

Let the number who entered by payment be $29341+x$,
" " ticket be $29341-x$,
so that the number was 58682 in all.

Hence $29341+x$ shillings were taken, and if the 7 odd shillings be deducted, there are $\frac{29334+x}{20}$ pounds.

$\therefore$ by the terms of the question,

$$\frac{29334+x}{20} = 29341-x-295,$$
$$= 29046-x,$$
$$\therefore 29334+x = 580920-20x,$$
$$21x = 551586,$$
$$x = 26266;$$

$\therefore$ 55607 persons entered by payment,
3075 " " ticket.

183. 11. Two toothed wheels work together and have a tooth in each marked that are together. When the smaller has turned round twice, its marked tooth is still 25 teeth from the tooth of the larger. After four turns it is 23 teeth beyond it. Find the number of teeth in each wheel.

Suppose that the large wheel has x teeth. Then $x-25$ of its teeth form a circular arc of the same length as two circumferences of the smaller wheel; while $x+23$ of its teeth make a circular arc of the same length as four circumferences of the smaller wheel. Hence $x+23$ must be double $x-25$;

$$\text{or } x+23 = 2(x-25),$$
$$= 2x-50,$$
$$x = 73,$$

the number of teeth of the large wheel.

Now $x-25$ or 48, is twice the number of teeth of the smaller wheel. Hence the smaller wheel has 24 teeth.

184. 12. At a public meeting a resolution was carried by a majority of 9, but if $\frac{1}{6}$ of those who voted for it had voted against it, it would have been lost by 3 votes. How many persons voted?

Suppose that $2x+9$ persons voted,
$x+9$ voting for the resolution,
x „ against „

Of these former suppose that $\frac{1}{6}(x+9)$ had voted on the contrary side. They leave

$\frac{5}{6}(x+9)$ voting for the resolution,
while $x+\frac{1}{6}(x+9)$ vote against it,

and now the latter number exceeds the former by 3;

$$\therefore\ x+\tfrac{1}{6}(x+9) = \tfrac{5}{6}(x+9)+3,$$
$$\therefore\ 6x+x+9 = 5x+45+18,$$
$$2x = 45+18-9 = 54,$$
$$\therefore\ 2x+9 = 63 \text{ is the number of persons who voted.}$$

185. The treatment of the following problem is an instance where two things appear to be required, yet are they so connected that one assumed unknown quantity suffices to determine both.

13. At an examination a candidate in order to pass was required to obtain a certain decimal of the whole number of marks allowed for the papers. A obtained by his marks ·35 which was not enough, and B obtained ·55 which was more than was necessary. If, however, 360 marks were added to A's total, and 120 marks taken from B's total,

A and B would each have just obtained enough to pass. Required the whole number of marks allotted to the papers, and the decimal of the whole required for a pass.

Let $100x$ marks be allotted to the papers,

$$\begin{aligned} &\text{then A obtains } 35x \text{ marks,} \\ &\quad ,, \quad \text{B} \quad ,, \quad 55x \quad ,, \end{aligned}$$

After the supposed alteration

$$\begin{aligned} &\text{A has } 35x+360 \text{ marks,} \\ &\text{B} \;,, \quad 55x-120 \quad ,, \end{aligned}$$

Now by the terms of the question these are equal, each being the exact number for a pass;

$$\begin{aligned} \therefore\ 55x-120 &= 35x+360, \\ 20x &= 120+360, \\ x &= 24; \end{aligned}$$

$\therefore$ 2400 marks are allotted to the papers,
and $35x+360 = 1200$, or $\cdot 5$ of the whole, suffices for passing.

Though this question has been made an example of the use of algebra, it can be solved by merely arithmetical means. For the altered marks of the candidates are together

$$\begin{aligned} &360-120+(\cdot 35+\cdot 55) \text{ of the whole,} \\ &\text{or } 240+\cdot 9 \text{ of the whole;} \end{aligned}$$

$\therefore$ the marks for a pass are

$$120+\cdot 45 \text{ of the whole,}$$

$$\therefore\ 120+\cdot 45 \text{ of the whole is } \cdot 55 \text{ of the whole}-120,$$

$$\begin{aligned} &\therefore\ \cdot 1 \text{ of the whole is } 240 \text{ marks,} \\ &\text{the whole is } 2400, \end{aligned}$$

and the number for pass is $120+\cdot 45$ of the whole $= 1200$.

186. 14. Two regular polygons are so related that the number of their sides is as 2 to 3, and the magnitude of their angles as 3 to 4. Find the figures.

Let one polygon have $2x$ and the other have $3x$ sides.

Since the interior angles of a polygon of $2x$ sides are

together $4x-4$ right angles (*Euclid*, I. 32), and since the polygon is regular, each of its angles is $\frac{4x-4}{2x}$ right angles. For the same reason each of the angles of the other polygon is $\frac{6x-4}{3x}$ right angles. Hence by the second condition of the question,

$$\frac{4x-4}{2x} = \frac{3}{4}\frac{6x-4}{3x},$$

$$\text{whence} \quad x = 2.$$

and the polygons have 4 and 6 sides, that is, they are a square and a regular hexagon.

187. 15. The present age of a person between 21 and 29 years old is such that 18 years hence his age will be expressed by the same digits in reversed order. How old is he?

Let him be $21+x$ years old, so that in the number expressing his age, 2 is the tens digit and $1+x$ the units digit.

After 18 years have elapsed he will be $39+x$ years old. Now this by the question is a number where $1+x$ is the tens digit and 2 is the units digit.

$$\therefore 39+x = (1+x)10+2,$$

$$x = 3,$$

or the person is 24 years old.

The following is a similar question :

16. The present age of a person between 40 and 50 years of age is such that 18 years ago it was expressed by the same digits in reversed order. How old is he?

Ans. His age is 42 years.

188. *Problems for Exercise.*

17. A and B have equal sums. If A had 15*s.* more, and B had 9*s.* less, A would have three times as much as B. What money have they? *Ans.* One guinea each.

18. A having three times as much money as B, gives B

150*l.* and then finds that he has only half as much again as B. How much had each at first?

Ans. A had 750*l.*, B 250*l.*

19. The sum of two numbers is double their difference. If the smaller of the two is 4, what is the larger? *Ans.* 12.

20. Out of a cask of wine $\frac{4}{5}$ full, 10 gallons are drawn, and the cask is then $\frac{2}{3}$ full. How much can it hold?

Ans. 75 gallons.

21. Find two numbers differing by 8, such that four times the less exceeds twice the greater by 10. *Ans.* 13 and 21.

22. A person buys a certain number of apples at the rate of 5 for 2*d.*, and sells half of them at 2 a penny, and the rest at 3 a penny, and gains a penny by the transaction. How many does he buy? *Ans.* 60 apples.

23. A person buys a certain number of apples at 2 a penny, and an equal number at 3 a penny, and selling them at 5 for 2*d.*, he loses a penny by the transaction. How many does he buy? *Ans.* He buys 30 apples at each rate.

24. A vessel containing some water was filled up by pouring in 42 gallons, and then there was in the vessel seven times as much water as at first. How many gallons did the vessel hold? *Ans.* 49 gallons.

25. A is twice as old as B, and in eleven years their ages will be as the numbers 5 and 3. What are their ages now?

Ans. A is 44 years of age and B is 22 years.

26. The difference between the 8th and 12th parts of a certain number exceeds by 1 the difference between the 9th and 15th parts of a number 108 less. Find the number.

Ans. 1368.

27. A farmer has two stacks of hay containing 54 loads in all. He uses 12 loads from the smaller of them, and it afterwards appears that the uncut stack contains twice as many loads as the remainder of the cut stack. Find the number of loads in each stack at first.

Ans. 28 and 26 loads.

28. A and B have the same income. A contracts an

annual debt amounting to $\frac{1}{7}$ of it; B lives on $\frac{4}{5}$ of it. At the end of 10 years B lends A money enough to pay off his debts, and then has 160*l.* to spare. What is the annual income of each? *Ans.* 280*l.*

29. Of a certain sum A receives 5*l.* and $\frac{1}{8}$ of the remainder; B receives 10*l.* and $\frac{1}{8}$ of what is then left; and C receives the balance, viz. 15*l.* Find the original sum.

Ans. 40*l.* 18*s.* 9*d.*

30. The difference of the squares of two consecutive numbers is 15. What are the numbers? *Ans.* 7 and 8.

31. From each of three pieces of cloth of equal length 19 yards were cut, and from another of double the length 34 yards. The remainders measured together 184 yards. What was the original length of each piece?

Ans. The longer one is 110 yards, the shorter ones 55 yards each.

32. A piece of stuff was cut in two, and the lengths of the portions were such that for every 5 yards of one there were 6 yards of the other. After 10 yards had been taken from each portion, the remainders were sold for 5*l.* and 6*l.* 10*s.*, respectively. Find the number of yards in the length of each portion, and also selling price per yard.

Ans. There were 30 and 36 yards, and the price was 5*s.* per yard.

33. A and B adventure equal sums in trade: A gains 100*l.*, and B loses so much that his money is now $\frac{2}{3}$ of A's money. If each gave the other $\frac{1}{3}$ of his present sum, B's loss would be diminished by $\frac{1}{2}$. What did each adventure?

Ans. 800*l.*

34. Of a farm, 5 acres more than $\frac{2}{3}$ are arable; 2 acres more than $\frac{4}{5}$ of the remainder are pasture; and there are besides 5 acres. Find the number of acres in the farm.

Ans. 120 acres.

35. 100 acres of land were bought for 4,220*l.*, part at 37*l.* an acre and the rest at 45*l.* an acre. How many acres were there of each kind?

Ans. 35 acres at 37*l.*, and 65 acres at 45*l.*

36. A person after paying an income-tax of 6*d.* in the pound gave away $\frac{1}{13}$ part of the remainder, and then had 540*l.* left. What was his original income?

Ans. The original income is 600*l.*

37. A person paid a bill of 3*l.* 14*s.* with shillings and half-crowns, giving 41 coins altogether. How many coins of each kind were there?

Ans. 19 shillings, 22 half-crowns.

189. 38. A man has a number of pennies which he tries to arrange in the form of a square. On the first attempt he has 130 over. When he increased the side of the square by 3 pennies he has only 31 over. How many pennies has he?

In the first attempt let there be x pennies in each row. Since the form is a square there must be also x rows, and therefore $x \times x$, or x^2, pennies in the square. Hence the whole number of pennies is x^2+130.

In the second arrangement the number of pennies in the square is $(x+3)^2$, and the whole number of pennies is $(x+3)^2+31$.

Hence,

$$\begin{aligned} x^2+130 &= (x+3)^2+31 \\ &= x^2+6x+9+31; \\ \therefore\ 6x &= 90, \\ x &= 15, \end{aligned}$$

and the number of pennies is $(15)^2+130$, or 355.

The method here given can be applied to the following problem:

39. A colonel wishes to arrange his men in a solid square. In the first formation he has 39 men over. When he increases the side of the square by 1 man he wants 50 men to complete the square. How many men has he?

Ans. 1975 men.

The following questions introduce percentages:

190. 40. A manufacturer adds to the cost price of goods 20 per cent. of it to give the selling price; afterwards to effect a rapid sale he deducts from the selling price of each

article a discount of 10 per cent., and then obtains on each article a profit of 8 shillings. What was the cost price of each article?

Let each article cost x shillings. The selling price at first adopted is $x+\frac{20}{100}x$ or $\frac{6x}{5}$ shillings. From this price $\frac{1}{10}$ is deducted leaving $\frac{9}{10}$, so that an article is finally sold for $\frac{9}{10}\cdot\frac{6x}{5}$ shillings or $\frac{27x}{25}$ shillings. By the question, this final price gives a profit of 8 shillings, or must be $x+8$ shillings.

$$\therefore\ \frac{27x}{25}=x+8,$$

$$\frac{2x}{25}=8,$$

$$x=100,$$

or the cost price of each article is $5l.$

191. 41. A person invests 14,970$l.$ in the purchase of 3 per cents. at 90 and $3\frac{1}{4}$ per cents. at 97. His total income being 500$l.$, how much of each stock did he buy?

Let him expend $7485+xl.$ in 3 per cents.,

$7485-xl.$ in $3\frac{1}{4}$ per cents.

In the former investment

$90l.$	produces	an annual	income of	$3l.$,
$1l.$	”	”	”	$\frac{1}{30}l.$,
$7485+xl.$	”	”	”	$\frac{7485+x}{30}l.$

In the latter investment

$97l.$	produces	an annual	income of	$\frac{13}{4}l.$,
$1l.$	”	”	”	$\frac{13}{4\times 97}l.$,
$7485-xl.$	”	”	”	$\frac{13(7485-x)}{4\times 97}l.$

Hence, his whole income of 500*l.* being the sum of these separate incomes,

$$500 = \frac{7485+x}{30} + \frac{13}{4 \times 97}(7485-x),$$
$$x = 1665;$$

$\therefore$ he expends 9150*l.* in 3 per cents.,
5820*l.* in $3\frac{1}{4}$ per cents.

192. 42. On a sum of money borrowed interest is to be paid at the rate of 5 per cent. per annum. After a time 600*l.* of the loan is paid off, and the interest on the remainder is now reduced to 4 per cent., and the yearly interest is by these combined causes lessened by one-third. What was the sum borrowed?

Let $x+3$ hundreds of pounds be borrowed, the interest on which is at first $5(x+3)$*l.* yearly.

After the repayment, $x-3$ hundreds of pounds remain at interest, and the interest thereon is $4(x-3)$*l.* yearly.

This latter interest is $\frac{2}{3}$ of the former by the condition of the problem,

$$\therefore \quad 4(x-3) = \tfrac{2}{3} \text{ of } 5(x+3),$$
$$\text{or } 12(x-3) = 10(x+3);$$
$$\therefore x = 33,$$
$$x+3 = 36,$$

or 3600*l.* was originally borrowed.

193. 43. A merchant lost a cargo at sea which he had insured. The broker offered him a sum of money for his loss, which the merchant refused as 10 per cent. below his estimated value of the loss. The broker then offered 379*l.* 15*s.* more than he offered at first, and the whole amount of the second offer was $5\frac{1}{2}$ per cent. in excess of the estimated value. What was that value, and what did the broker first offer?

For the purpose of avoiding fractions it may be convenient to assume the estimated value of the loss to be $10x$ pounds.

On this 10 per cent. is x, so that the broker's first offer is $9xl$.

The second offer is $9x+379\frac{3}{4}l$.

This exceeds the estimated value by $379\frac{3}{4}-xl$.

But $5\frac{1}{2}$ per cent. on the estimated value is $\frac{11x}{20}l$.

$$\text{Wherefore } 379\tfrac{3}{4}-x = \frac{11x}{20},$$

$$\frac{31x}{20} = 379\tfrac{3}{4},$$

$$x = \tfrac{20}{31}\times 379\tfrac{3}{4}l. = 245l.$$

$\therefore$ $10x$, the estimated value, is $2450l.$, and $9x$, the broker's first offer, is $2205l$.

194. 44. A and B join capital for a commercial enterprise, B contributing $250l.$ more than A. If their profits amount to 10 per cent. on their joint capital, B's share of them is 12 per cent. on A's capital. How much does each contribute?

If their profits are 10 per cent. on capital, or a tenth part of it, when they are divided according to capital advanced, each receives as his share a tenth of the capital he advances.

Let A advance $10xl$.

$\therefore$ B advances $10x+250l$.

$\therefore$ B's share of profit is $x+25l.$, and this is $\frac{12}{100}$ of A's capital.

$$\therefore\ x+25 = \tfrac{12}{100}10x = \frac{12x}{10} = \frac{6x}{5}$$

$$\frac{x}{5} = 25,$$

$$x = 125.$$

$\therefore$ A advances $10x$, or $1250l$.

B " " $1500l$.

195. 45. A grocer bought 200 pounds of tea and 1000 pounds of sugar, the price of the sugar being $\frac{1}{6}$ of that of the tea. He sold the tea at a profit of 40 per cent. and the sugar at a loss of $2\frac{1}{2}$ per cent., gaining on the whole 9*l.* 9*s.* 7*d.* What were his buying and selling prices?

Let the sugar be bought at x pence the pound.

$\therefore$ the tea „ $6x$ „ „

Selling price of the tea is $\frac{140}{100}\,6x$, or $\dfrac{42x}{5}$ pence the pound.

„ „ „ sugar is $\dfrac{97\frac{1}{2}}{100}x$, or $\frac{39}{40}x$ „ „

$\therefore$ gain on 1 lb. of tea is $\dfrac{12x}{5}$ pence.

„ 200 lbs. „ $12 \times 40x = 480x$ pence.

Loss on 1 lb. of sugar is $\dfrac{x}{40}$ pence,

„ 1000 lbs. „ $25x$ pence.

$\therefore$ $480x - 25x$, the whole gain in pence is the number of pence in 9*l.* 9*s.* 7*d.*,

$$\text{or } 455x = 2275,$$
$$\therefore x = 5.$$

Hence the buying price of tea is 30*d.*, and the selling price 42*d.* the pound.

The buying price of sugar is 5*d.*, and the selling price $4\frac{7}{8}$*d.* the pound.

196. 46. 1000 quills are bought for a sovereign. Half are sold at half-a-crown the hundred. At what price must the rest be sold to make 50 per cent. profit on the whole?

Ans. 3*s.* 6*d.* the hundred.

47. The cost price of each copy of a newspaper is $3\frac{1}{4}$*d.* The sum for which it is sold is 3*d.* The number of pence received for the insertion of advertisements is 30 per cent. of the number of copies issued beyond 10,000. Find the least number of copies that can be issued without loss.

Ans. 60,000 copies.

197. 48. In a mixture of wine and water there are two gallons of wine for every three gallons of water. If a gallon of wine be added the mixture becomes half wine and half water. Of how many gallons did the mixture consist?

In consistence with the statement of the problem we may suppose that there were originally $2x$ gallons of wine and $3x$ gallons of water forming the mixture, so that it is $5x$ gallons of mixture.

Obs.—Since x is open to be an integer or a fraction, this supposition does not restrict the mixture to consist of an exact number of gallons.

When the gallon of wine is added there are $2x+1$ gallons of wine and $3x$ gallons of water. By the question, these quantities are equal.

$$\therefore\ 3x = 2x + 1,$$
$$x = 1,$$

or the mixture originally consisted of 5 gallons, 2 being wine and 3 water.

198. 49. How much water must be mixed with 60 gallons of spirit which cost 1*l.* the gallon, that on selling the mixture at 22*s.* the gallon a gain of 17*l.* may be made?

Let x gallons of water, supposed valueless, be added.

Hence $60+x$ gallons of mixture, costing 60*l.*, will be sold for $\frac{11}{10}(60+x)$*l.*, and the gain is 17*l.*

$$\therefore\ \tfrac{11}{10}(60+x) - 60 = 17,$$
$$x = 10.$$

199. The following problems are similar:

50. How much water must be mixed with 80 gallons of spirit bought at 15*s.* the gallon, so that on selling the mixture at 12*s.* the gallon there may be a profit of 10 per cent. on the outlay? *Ans.* 30 gallons of water.

51. A cask of 110 gallons of wine is bought for 120*l.* What water must be mixed so that when the mixture is sold at a guinea the gallon 5 per cent. profit may be made?

Ans. 10 gallons of water.

200. 52. A brewer has a certain quantity of beer worth 1*s*. 6*d*. the gallon, and twice as great a quantity worth 1*s*. the gallon. They are mixed and sold at 1*s*. 3*d*. the gallon, and the profit is the price of 108 gallons of the mixture. How much has he of each?

Suppose that there are x gallons worth 1*s*. 6*d*. the gallon, and consequently $2x$ gallons worth 1*s*. the gallon. The former quantity has the value $\frac{3}{2}x$ shillings and the latter $2x$ shillings, making together $\frac{7}{2}x$ shillings.

There are $3x$ gallons of mixture which at $\frac{5}{4}$ shillings the gallon produce $\frac{15}{4}x$ shillings.

The profit then is $\frac{15x}{4} - \frac{7x}{2} = \frac{x}{4}$ shillings.

$$\therefore \frac{x}{4} = 108 \times \frac{5}{4},$$

$$x = 540.$$

There are then 540 gallons of the former, and 1080 of the latter kind of beer.

The following problem is a similar one:

53. A wine merchant has a certain quantity of sherry worth 30*s*. the gallon, and twice as great a quantity worth 20*s*. the gallon. He mixes them and sells the mixture at 25*s*. the gallon, and the profit is the price of 36 gallons of the mixture. How much has he of each wine?

Ans. 180 gallons of one, and 360 gallons of the other.

201. 54. There are two vessels, the smaller of which holds 21 gallons less than the larger. Both being originally full, from the larger were drawn off 4 gallons more than half its contents, and into it were transferred $1\frac{1}{2}$ gallons less than half the contents of the smaller; then 4 gallons were drawn from the contents of the smaller cask, $4\frac{1}{2}$ gallons added to the contents of the larger. The latter now contains $2\frac{1}{2}$ times as much as the former. Find the capacity of each cask.

Ans. 84 and 63 gallons.

202. Problems relating to speed and time of motion, and distances travelled, when the speed is invariable, generally require the following considerations. If an hour is made the unit of time, the speed is expressed as a certain number of miles an hour, and the distance travelled in any number of hours is the speed multiplied by this number. If, for instance, the speed is x miles an hour, the distance travelled in 5 hours is $5x$ miles, in 8 hours is $8x$ miles, and so on. The number of hours of travelling is given by dividing the distance travelled by the speed, or number of miles travelled in each hour. Thus the time of travelling 50 miles is $\frac{50}{x}$ hours. Again, the speed results from dividing the distance travelled by the number of hours taken to travel it. If x feet, for instance, are travelled over in 7 minutes the speed must be $\frac{x}{7}$ feet a minute.

203. 55. A man starts from home to ride to a place which, at the speed he usually maintains, he will reach in 2 hours, but when he has advanced 4 miles on his journey he has to return home. He starts again without delay, and now rides half as fast again as before, and reaches his destination 20 minutes later than he would have reached it if he had not turned back. What is the distance of the place to which he was going?

Let him usually ride $2x$ miles an hour, so that the place to which he is going, which he might reach in 2 hours, is $4x$ miles off.

Now he first rides 4 miles out and 4 miles back, or 8 miles, at the rate of $2x$ miles an hour, and therefore occupies the time $\frac{8}{2x}$ hours (202).

He then adopts the speed of $3x$ miles an hour, and rides $4x$ miles at this rate, occupying therefore $\frac{4x}{3x}$ or $\frac{4}{3}$ of an hour.

His whole time of travelling, being 20 minutes longer than if he had not turned back, is $2\frac{1}{3}$ hours.

$$\therefore \frac{8}{2x}+\frac{4}{3} = 2\frac{1}{3} = \frac{7}{3},$$

$$x = 4,$$

and $4x$, the distance required, is 16 miles.

204. 56. A man can row 6 miles in an hour with the stream of a river, and 4 miles in an hour against the stream, his speed in each direction being uniform. How far may he go in order that the time between leaving and returning to the place from which he started may be $2\frac{1}{2}$ hours?

Let x miles be the distance to which he goes from the starting-place.

It is immaterial whether the stream is in his favour in going or in returning, because in either case he has to row x miles with stream and x miles against stream, and will therefore in either case take the same time.

If he can row 6 miles with the stream in 1 hour, he can row 1 mile in $\frac{1}{6}$ hour, and x miles in $\frac{x}{6}$ hours. Similarly he can row x miles against the stream in $\frac{x}{4}$ hours. Now the whole time of rowing is the sum of these separate times.

$$\therefore \frac{x}{6}+\frac{x}{4} = \frac{5}{2},$$

$$x = 6,$$

and the distance the rower goes out is 6 miles.

205. The following is a similar problem:

57. A person walked to the top of a mountain at the uniform rate of $2\frac{1}{3}$ miles an hour, and down again by the same way at the uniform rate of $3\frac{1}{2}$ miles an hour. He was out 5 hours. How far did he walk?

Ans. 7 miles up and 7 miles down.

206. 58. In a mile race the uniform speeds of two runners are proportioned to 11 and 8. The latter, having 320 yards start, is beaten by half a minute. What are their speeds?

The faster runner runs 1760 yards, the whole mile, and the slower runner runs 1440 yards. Let one run $11x$ yards, and the other run $8x$ yards in a minute. The former runs 1760 yards in $\frac{1760}{11x}$ or $\frac{160}{x}$ minutes (202), and the latter runs 1440 yards in $\frac{1440}{8x}$ or $\frac{180}{x}$ minutes. Hence by the terms of the question,

$$\frac{180}{x} - \frac{160}{x} = \frac{1}{2},$$

$$x = 40.$$

The faster runner runs 440 yards in a minute, the slower runs 320 yards in a minute.

59. Suppose their speeds were as 5 and 4, and the latter having half a minute start is beaten by 176 yards. What are their speeds? *Ans.* 440 and 352 yards in a minute.

207. 60. AB is a railway from A to B, 200 miles long. Three trains P, Q, R travel upon it at uniform rates, 25, 20, and 30 miles an hour respectively. P and Q leave A at 7 A.M. and 10.15 A.M. respectively. R leaves B at 11.30 A.M. When will R be equidistant from P and Q.

The position of the trains will be expressed in terms of the time of day when they are considered. Let R be in the proposed position, equidistant between the other trains at x hours after the midnight which preceded their starting.

Then P has been running $x-7$ hours, and is $25(x-7)$ miles from A.

Q has been running $x-10\frac{1}{4}$ hours, and is $20(x-10\frac{1}{4})$ miles from A.

R has been running $x-11\frac{1}{2}$ hours, and is $30(x-11\frac{1}{2})$ miles from B; consequently $200-30(x-11\frac{1}{2})$ miles from A.

Hence the distance between P and R is

$$20(x-10\tfrac{1}{4})-200+30(x-11\tfrac{1}{2}),$$
$$\text{or } 50x-750 \text{ miles};$$

and the distance between R and Q is

$$200-30(x-11\tfrac{1}{2})-25(x-7)$$
$$\text{or } 720-55x \text{ miles}.$$

Now by the condition of the question,

$$50x-750 = 720-55x,$$
$$105x = 1470,$$
$$x = 14;$$

or the time required is 2 o'clock, P.M.

61. Suppose the railway 220 miles long, and the times of starting of P and Q to be 7 and 8.15, while that of R is 10.30. When is P equidistant from Q and R?

Ans. At noon.

208. 62. A traveller by a railway, on which the telegraph posts are 30 in a mile, observed that between two points on the line he uniformly passed 10 posts in a minute. Having noticed the distance between these points, and calculated the speed of the train on supposition of 20 posts in a mile, he found that the train is 5 minutes more in accomplishing the distance than he expected. Find the distance between the two points.

Let the distance be $\frac{x}{30}$ miles.

Then the traveller passes x posts, and as he passes 10 posts in a minute he is $\frac{x}{10}$ minutes in passing between the two points.

But by his mistake in the number of posts in a mile he considers that he passes over $\frac{1}{20}$ of a mile in $\frac{1}{10}$ of a minute, or a mile in 2 minutes, and $\frac{x}{30}$ miles in $\frac{x}{15}$ minutes. Be

tween this calculated time and the true time there is a difference of 5 minutes, or,

$$\frac{x}{10} - \frac{x}{15} = 5,$$
$$x = 150,$$

and $\frac{x}{30}$, the distance of the points, is 5 miles.

209. 63. The termini of a railway 126 miles long are A and C, and the station B, at which a certain train stops 15 minutes, is 70 miles from A. The whole journey from A to C takes 15 minutes less than twice as long as the journey from A to B. Determine the average rate of the train, including all stoppages except that at B.

Ans. Average rate of train 28 miles per hour.

64. A train started from London at 6.30 for Dover, where it was due at 10.30. After proceeding half-way at the ordinary uniform rate it was detained three quarters of an hour. The speed was then increased by 8 miles an hour, and the train arrived at 10.43. What was the distance travelled and the usual speed of the train?

Ans. The distance was 88 miles, the speed 22 miles an hour.

65. Two clocks, A and B, go very badly, A gaining 15 minutes (i.e. 15 minute-divisions) every hour, B gaining 3 minute-divisions every hour. A was set to the right time at 12 o'clock, when B was 21 minutes fast. At this moment B is $2\frac{3}{10}$ times as fast as A. What o'clock is it now by a correct watch? *Ans.* 40 minutes past 12.

66. A and B have each the same quantity of work to do. A began to work 2 hours before B, but afterwards they worked and rested together till B had finished his work, which was 1 hour before A had finished his. If, when A had done half his work, each had taken the other's remaining work, B would have finished 2 hours and $1\frac{1}{2}$ minutes before A. How long did each take to do the work? *Ans.* 63 and 60 hours.

CHAPTER V.

SIMULTANEOUS EQUATIONS.

210. When an equation contains two unknown quantities, if no other condition between these quantities is known, they cannot be determined, though many consistent pairs of values of them may be found for them.

$$\begin{aligned} \text{Thus if}\quad & 2x+3y = 25, \\ \text{let}\quad & y = 1, \text{ then } x = 11\,; \\ & y = 2, \quad ,, \quad x = 9\tfrac{1}{2}\,; \\ & y = 3, \quad ,, \quad x = 8\,; \end{aligned}$$

and this process continued will give for any value, integral or fractional, positive or negative, which we like to assign to y, a corresponding value of x. So if we assign at pleasure values to x, corresponding values of y result. Thus, in this instance, a single equation in x and y gives an unlimited number of pairs of values of those quantities, each pair satisfying the equation.

If, however, we have also another equation in x and y, as, for instance, $x-2y = 2$, to be satisfied by the same values of x and y as the former; then while, as in the former instance, it will be possible to find an unlimited number of solutions of this equation too, yet there is only one pair, viz., $x = 8$, $y = 3$, which will satisfy both equations at once. Hence, when two distinct and compatible equations in x and y are presented, we have no longer an unlimited number of pairs of values admissible, but only a limited number of solutions can exist. Equations satisfied by the same values of the unknown quantities are called *simultaneous equations.*

211. *Obs.*—The two equations above mentioned have been described as distinct and compatible.

If the pair of equations were

$$\left.\begin{aligned} 3x+5y &= 7 \\ 6x+10y &= 14 \end{aligned}\right\},$$

though they may appear different in form, they are not distinct, since if the second be divided throughout by 2 it produces the first, and consequently states no property respecting x and y beyond that which the former equation embodies.

If, again, the pair of equations were

$$\begin{aligned} 3x+5y &= 7, \\ 6x+10y &= 15, \end{aligned}$$

they are not compatible, because if the former were multiplied throughout by 2 it gives $6x+10y=14$, while the latter asserts that $6x+10y=15$. No finite values of x and y therefore can satisfy these two equations at the same time.

212. When two equations are presented containing two unknown quantities, x and y for instance, equations distinct and compatible, it is possible so to combine them that first one of the unknown quantities and then the other may disappear.

By combination, for instance, of the two in a proper manner a single equation may result containing only x, and therefore serving to determine x. In this case the quantity y is said to have been 'eliminated' between the equations. Or again, x may by proper processes be eliminated, and an equation obtained containing only y, and sufficient for determining y.

213. It has been seen that when the same quantity has been added to or subtracted from both members of an equation, the results form the members of an equation (134). So, also, if of any two equal quantities one be added to the former and the other to the latter member of an equa-

tion, or if any equal quantities be subtracted one from the former and the other from the latter member, then also the two results form members of an equation. If, again, the members of an equation be multiplied or divided by equal quantities, the results are members of an equation, just as when the members of an equation are multiplied by or divided by the same quantity.

214. On these principles it is generally possible to eliminate between two equations one of the two unknown quantities which they contain, and then to determine the other. When one is determined, then if its value be substituted in either of the given equations, that equation suffices to determine the other unknown quantity. This is what is generally possible, the exception being when algebraical processes arise which cannot be effected.

215. Let the equations be

$$\left.\begin{array}{l} 2x+3y=7 \\ 2x-3y=1 \end{array}\right\}.$$

Since the second states that $2x-3y$ and 1 are equal quantities, let them be added respectively to the first and second members of the first equation,

$$\therefore\ 4x=8.$$

Thus y has been eliminated and an equation arises qualified to determine x, for division by 4 at once gives $x=2$. Then if 2 be substituted for x in either of the original equations, that equation gives $y=1$.

Again, if the quantities $2x-3y$ and 1, which the second equation declares to be equal, were respectively subtracted from the members of the first equation,

$$6y=6,$$

and an equation arises from which x has been eliminated, and the value $y=1$ is found. Then, by substitution in either of the original equations, $x=2$.

216. Let
$$\left.\begin{aligned} x^2-y^2 &= 8y \\ x+y &= 4 \end{aligned}\right\}.$$

Since the second equation asserts that $x+y$ is equal to 4, let the members of the first equation be respectively divided by these equal quantities, and we have

$$x-y = 2y.$$

Subtract these equal quantities from the respective members of the second equation, and then

$$2y = 4-2y,$$

an equation from which x is eliminated; and by transposition,

$$4y = 4,$$
$$y = 1.$$

Then $$x = 4-y = 3.$$

217. These instances may suffice to show what is the general object to be aimed at in order to effect the solution of two simultaneous equations. It is to remove or eliminate one of the two unknown quantities, and thus to determine them in succession. The artifices required for this object are of infinite variety, to be adopted as the occasion requires them, and only to be learned by the study of examples and by practice.

218. Let
$$ax+by = c,$$
$$bx-ay = d,$$

a, b, c, d being known quantities in terms of which x and y have to be expressed, and thus to become also known.

If the first equation be multiplied by a and the second by b, the results are

$$a^2x+aby = ac$$
$$b^2x-aby = bd.$$

Now of these new equations let the second be added to the first, member by member,

$$\therefore\ a^2x+b^2x = ac+bd,$$

and y has been eliminated.

$$\therefore\ (a^2+b^2)x = ac+bd,$$
$$x = \frac{ac+bd}{a^2+b^2}.$$

Then from either of the original equations y can be found.

From the former of these,

$$by = c-ax,$$
$$= c-a\frac{ac+bd}{a^2+b^2},$$
$$= \frac{b^2c-abd}{a^2+b^2},$$
$$\therefore\ y = \frac{bc-ad}{a^2+b^2}.$$

It would have been equally possible to eliminate x by multiplying each equation so that x should have the same coefficient in each, and then disappear in subtraction. Let the equations be multiplied, the first by b, the second by a.

$$\therefore\ abx+b^2y = bc,$$
$$abx-a^2y = ad.$$

Then by subtraction,

$$b^2y+a^2y = bc-ad,$$
$$y = \frac{bc-ad}{b^2+a^2}, \text{ as before.}$$

219. There is a neat and concise notation whereby symbols are used instead of words to describe the process of multiplying or dividing an equation by any quantity throughout, or combining by addition or subtraction the members of two equations. The equations being marked by numbers in brackets so that (1) (2), for instance, mean the equations to which they are affixed, $(1)+(2)$ means the addition of these equations member to member; $(1)-(2)$ means the subtracting of the second from the former, member from member; $(1)\times a$ means the result of multiplying the equation (1) by some quantity a throughout.

Thus the solution of the last example may be written in the following more compendious form:

$$\left.\begin{aligned} ax+by &= c \quad (1),\\ bx-ay &= d \quad (2), \end{aligned}\right\}$$

$$\begin{aligned} (1)\times a \text{ gives } a^2x+aby &= ac \ (3),\\ (2)\times b \quad ,, \quad b^2x-aby &= bd \ (4),\\ (3)+(4) \quad ,, \quad a^2x+b^2x &= ac+bd;\\ \therefore\ x &= \frac{ac+bd}{a^2+b^2}. \end{aligned}$$

Then from (1) $by = c-\dfrac{a^2c+abd}{a^2+b^2} = \dfrac{b^2c-abd}{a^2+b^2}$,

$$y = \frac{bc-ad}{a^2+b^2}.$$

220.
$$\left.\begin{aligned} \frac{x}{9}+\frac{y}{5} &= 11\\ \frac{x}{4}+\frac{y}{7} &= 14 \end{aligned}\right\}.$$

These equations are first to be cleared of fractions by multiplying the first throughout by 45 and the second by 28, and they give

$$\left.\begin{aligned} 5x+9y &= 495 \quad (1),\\ 7x+4y &= 392 \quad (2). \end{aligned}\right\}$$

$$\begin{aligned} (1)\times 7 \text{ gives } 35x+63y &= 3465,\\ (2)\times 5 \quad ,, \quad 35x+20y &= 1960; \end{aligned}$$

$\therefore$ if the latter be subtracted from the former,

$$43y = 1505,$$
$$y = 35.$$

Then from (1),

$$5x = 495-9\times 35,$$
$$x = 36.$$

In instances like these the object aimed at is to bring one of the unknown quantities to have the same coefficient in both equations, that it may thus be eliminated by subtraction.

221.
$$\left.\begin{aligned} x(y+3) &= y(x+7) \\ 4y+14 &= 5x-9 \end{aligned}\right\}.$$

When the multiplications expressed in the former equation are effected,

$$\begin{aligned} xy+3x &= xy+7y, \\ \text{or } 3x &= 7y, \\ \text{or } x &= \frac{7y}{3}. \end{aligned}$$

Then in the second equation,

$$\begin{aligned} 4y &= 5x-9-14, \\ &= \frac{35y}{3}-23, \\ 12y &= 35y-69, \\ 35y-12y &= 69, \\ 23y &= 69, \\ y &= 3, \end{aligned}$$

$$\therefore x = \frac{7y}{3} = 7.$$

In this example the first equation gives at once x in terms of y, and substitution for x in the second equation is then the easiest means of finding y.

222.
$$\left.\begin{aligned} 7x+4y &= 17y, \quad (1) \\ 6x-10y &= 8. \quad (2) \end{aligned}\right\}$$

By transposition (1) becomes,

$$\begin{aligned} 7x &= 13y, \\ \text{or } x &= \tfrac{13}{7}y, \\ \text{or } 6x &= \tfrac{78}{7}y; \end{aligned}$$

then equation (2) gives,

$$\frac{78x}{7} - 10y = 8,$$

$$78y - 70y = 56,$$

$$8y = 56;$$

$$\therefore y = 7,$$

$$\therefore x = 13.$$

223.

$$\left.\begin{aligned} \frac{3}{x} - \frac{2}{y} &= \frac{1}{a}, \quad (1) \\ \frac{2}{x} - \frac{1}{y} &= \frac{2}{a}. \quad (2) \end{aligned}\right\}$$

To solve these equations we shall aim at finding $\frac{1}{x}$ and $\frac{1}{y}$ and then x and y become at once known.

$(1) - 2 \times (2)$ gives

$$-\frac{1}{x} = -\frac{3}{a},$$

$$\therefore x = \frac{a}{3};$$

$\therefore$ from (2),

$$\frac{1}{y} = \frac{2}{x} - \frac{2}{a},$$

$$= \frac{4}{a},$$

$$y = \frac{a}{4}.$$

The following equations have a similar form:

$$\left.\begin{aligned} \frac{a}{x} + \frac{b}{y} &= \frac{1}{a} + \frac{1}{b}, \\ \frac{b}{x} - \frac{a}{y} &= \frac{1}{a} - \frac{1}{b}. \end{aligned}\right\}$$

where $x = y = ab$.

224. *Examples for Practice.*

1. $\left.\begin{array}{r} 9x-4y = 8 \\ 5x+3y = 41 \end{array}\right\}.$ *Ans.* $x=4$, $y=7$.

2. $\left.\begin{array}{r} 5x-7y = 20 \\ 9x-11y = 44 \end{array}\right\}.$ *Ans.* $x=11$, $y=5$.

3. $\left.\begin{array}{r} 4x-y = 5 \\ 6y-5x = 8 \end{array}\right\}.$ *Ans.* $x=2$, $y=3$.

4. $\left.\begin{array}{r} 3x-5y = 13 \\ 2x+7y = 81 \end{array}\right\}.$ *Ans.* $x=16$, $y=7$.

5. $\left.\begin{array}{r} x+2y = 15 \\ 5x-19y = 17 \end{array}\right\}.$ *Ans.* $x=11$, $y=2$.

6. $\left.\begin{array}{r} 5x+11y = 146 \\ 11x+5y = 110 \end{array}\right\}.$ *Ans.* $x=5$, $y=11$.

7. $\left.\begin{array}{r} 3x+\dfrac{y}{3} = 36 \\ 6y-2x = 32 \end{array}\right\}.$ *Ans.* $x=11$, $y=9$.

8. $\left.\begin{array}{r} 2x-\dfrac{y-3}{5} = 4 \\ 3y+\dfrac{x-2}{3} = 9 \end{array}\right\}.$ *Ans.* $x=2$, $y=3$.

9. $\left.\begin{array}{r} \frac{1}{2}(x+y) = \frac{1}{5}(2x+10) \\ \frac{1}{4}(3x-y) = 2y+3 \end{array}\right\}.$ *Ans.* $x=10$, $y=2$.

10. $\left.\begin{array}{r} x-\frac{1}{7}(y-2) = 5 \\ 4y-\frac{1}{3}(x+10) = 3 \end{array}\right\}.$ *Ans.* $x=5$, $y=2$.

11. $\left.\begin{array}{r} \dfrac{4x+5y}{40} = x-y \\ \dfrac{2x-y}{3}+2y = \frac{1}{2} \end{array}\right\}.$ *Ans.* $y=\frac{1}{6}$, $x=\frac{1}{4}$.

12. $\left.\begin{array}{r} x-\dfrac{y-2}{7} = 5 \\ 4y-\dfrac{x+10}{3} = 3 \end{array}\right\}.$ *Ans.* $x=5$, $y=2$.

13. $$\left.\begin{aligned} \frac{y+3}{5}+\frac{y-x}{6} &= 2x-8 \\ \frac{2x-y}{7}+3x &= 2y-6 \end{aligned}\right\}.$$ *Ans.* $x=6$, $y=12$.

14. $$\left.\begin{aligned} \frac{x}{2}-y &= 1 \\ x-\frac{y}{2} &= 8 \end{aligned}\right\}.$$ *Ans.* $x=10$, $y=4$.

15. $$\left.\begin{aligned} \frac{x+2}{7} &= \frac{7}{8} \\ \frac{x}{y-2} &= \frac{5}{6} \end{aligned}\right\}.$$ *Ans.* $x=5$, $y=8$.

16. $$\left.\begin{aligned} \frac{7x+3y}{11}-\frac{4x-5y}{5} &= \frac{15-y}{4} \\ 5y &= 3x \end{aligned}\right\}.$$ *Ans.* $x=5$, $y=3$.

17. $$\left.\begin{aligned} \frac{5x-3y}{10}-\frac{3x-2y}{11} &= \frac{x-y}{2} \\ x+7 &= 2(y+2) \end{aligned}\right\}.$$ *Ans.* $x=7$, $y=5$.

18. $$\left.\begin{aligned} (x+5)(y+7) &= (x+1)(y-9)+112 \\ 2x+10 &= 3y+1 \end{aligned}\right\}.$$

Ans. $x=3$, $y=5$.

19. $$\left.\begin{aligned} \frac{4x+3}{5}-\frac{y+7}{4} &= \tfrac{1}{12}\left(\frac{x}{3}-\frac{y}{5}\right) \\ 3(x+y)-10(y-x) &= \frac{7y-3x}{6\frac{1}{2}} \end{aligned}\right\}.$$

Ans. $x=3$, $y=5$.

20. $$\left.\begin{aligned} \frac{3x+4y}{8}+\frac{5y+7x}{3} &= \frac{x+5y}{12}+\frac{y-2x}{24}+8\tfrac{7}{24} \\ 2y-x &= 3(4x-y)+13 \end{aligned}\right\}.$$

Ans. $x=\frac{7}{13}$, $y=4$.

21. $$\left.\begin{aligned} \frac{1}{x}+\frac{1}{y} &= \frac{1}{a} \\ \frac{1}{x}-\frac{1}{y} &= \frac{1}{b} \end{aligned}\right\}.$$ *Ans.* $x=\dfrac{2ab}{a+b}$, $y=-\dfrac{2ab}{a-b}$.

22. $\left.\begin{aligned}\frac{x}{a}+\frac{y}{b}=\frac{c}{d}\\ \frac{x}{e}-\frac{y}{f}=\frac{g}{h}\end{aligned}\right\}.$ *Ans.* $x=\frac{ae}{dh}\cdot\frac{bch+fgd}{be+af}$, $y=\frac{bf}{dh}\cdot\frac{ach-deg}{be+af}$.

23. $\left.\begin{aligned}ax+by&=c^2\\ \frac{a}{b+y}-\frac{c}{a+x}&=0\end{aligned}\right\}.$

Ans. $x=\frac{(b^2+c^2)c-a^2b}{a(b+c)}$, $y=\frac{a^2+c^2-bc}{b+c}$.

225. Let the equations proposed be

$$3x+5y-70=\frac{x}{5}+\frac{8y}{3}=x+y+8.$$

Three quantities being here given as equal to one another, we can form two equations among them by pairing them in either of two ways. Thus we may have

$$\left.\begin{aligned}3x+5y-70&=x+y+8\\ \frac{x}{5}+\frac{8x}{3}\quad&=x+y+8\end{aligned}\right\}$$

as one pair of equations: or else

$$\left.\begin{aligned}3x+5y-70&=x+y+8\\ 3x+5y-70&=\frac{x}{5}+\frac{8y}{3}\end{aligned}\right\}.$$

If we adopt the former arrangement, the equations become

$$2x+4y=78,$$

$$\text{or}\qquad x+2y=39, \qquad (1)$$

$$\text{and } 12x-25y=-120. \quad (2)$$

$$(1)\times 12 \text{ gives } 12x+24y=468;$$

$$\therefore\ 49y=588,$$

$$y=12,$$

$$x=39-2y=39-24=15.$$

226. $$\frac{x-a}{y+b}=\frac{y-b}{x+a}=\frac{c}{d}.$$

Three quantities being given as equal to one another, we have the option of forming a pair of equations in two different ways. If we take the pair

$$\left.\begin{aligned} \frac{x-a}{y+b}&=\frac{c}{d}, \quad (1)\\ \frac{y-b}{x+a}&=\frac{c}{d}. \quad (2)\end{aligned}\right\}$$

these equations are

$$\left.\begin{aligned} dx-cy &= ad+bc, \quad (1)\\ cx-dy &= -ac-bd. \quad (2)\end{aligned}\right\}$$

$(1).d-(2).c$ gives

$$(d^2-c^2)x = ad^2+ac^2+2bcd,$$
$$x=\frac{ad^2+ac^2+2bcd}{d^2-c^2}.$$

$(1).c-(2).d$ gives

$$(d^2-c^2)y = 2acd+bc^2+bd^2,$$
$$y=\frac{bc^2+bd^2+2acd}{d^2-c^2}.$$

227. If three distinct and compatible equations are presented between three unknown quantities, one of them has usually to be eliminated, so as to produce two equations between the remaining two, and these two quantities being determined by the methods already given, the one which was first eliminated can then also be found, and the solution thus completed.

Ex.
$$\left.\begin{aligned} x+2y-3z &= -4, \quad (1)\\ x+2z-\ y &= 9, \quad (2)\\ 2y+2z-\ x &= 15. \quad (3)\end{aligned}\right\}$$

It will be most convenient to eliminate x.

$(1)+(3)$ gives $4y-\ z = 11$, (5)

$(2)+(3)$ „ $y+4z = 24$, (6)

$4\times(5)+(6)$ „ $17y = 68, y=4.$

Then from (5), $z = 4y - 11 = 5$,
and from (1), $x = 3z - 2y - 4 = 3$.

228. In the following example it happens that the solution can be effected without that successive elimination of unknown quantities, which has been made in the last examples.

$$\left.\begin{aligned} x+y-z &= 3, \quad (1) \\ x+z-y &= 5, \quad (2) \\ y+z-x &= 7. \quad (3) \end{aligned}\right\}$$

(1) + (2) + (3) gives

$$x+y+z = 15, \quad (4)$$

$$\begin{aligned} (4)-(3) \text{ gives } 2x &= 8, & x &= 4, \\ (4)-(2) \quad ,, \quad 2y &= 10, & y &= 5, \\ (4)-(1) \quad ,, \quad 2z &= 12, & z &= 6. \end{aligned}$$

229. *Examples for Practice.*

1. $\left.\begin{aligned} y+z &= 7 \\ x+z &= 9 \\ x+y &= 10 \end{aligned}\right\}.$ *Ans.* $x = 6$, $y = 4$, $z = 3$.

2. $\left.\begin{aligned} 2x+4y-3z &= 22 \\ 4x-2y+5z &= 18 \\ 6x+7y-z &= 63 \end{aligned}\right\}.$ *Ans.* $y = 7$, $x = 3$, $z = 4$.

3. $z = \dfrac{9-4x+3y}{2} = \dfrac{2x+5y-4}{3} = \dfrac{5x+6y-18}{2}.$
Ans. $x = 2$, $y = 3$, $z = 5$.

4. $\left.\begin{aligned} \frac{yz}{y+z} &= 5\tfrac{1}{7} \\ \frac{xz}{x+z} &= 4 \\ \frac{xy}{x+y} &= 3\tfrac{3}{5} \end{aligned}\right\}.$ *Ans.* $x = 6$, $y = 9$, $z = 12$.

5. $\left.\begin{aligned} xy &= 3(x+y) \\ xz &= 8(x+z) \\ 7yz &= 9(y+z) \end{aligned}\right\}.$ *Ans.* $x = -\frac{144}{23}$, $y = \frac{144}{71}$, $z = \frac{144}{41}$.

CHAPTER VI.

PROBLEMS PRODUCING SIMULTANEOUS EQUATIONS.

230. Simultaneous simple equations will now be employed in solving problems. We gain, it may be anticipated, increased power in expressing, algebraically, the facts of a problem, when we can introduce more than one unknown quantity to assist us in representing the elements to be determined.

231. It has been seen (210) that we can find values for two unknown quantities when we have two distinct and compatible equations between them. So, also, we can find values for three unknown quantities, if we have three distinct and compatible equations among them. Hence, generally, as many symbols as are assumed to designate unknown quantities in a problem, so many distinct and compatible equations must we obtain from the conditions of the problem, if the solution is to be effected.

232. 1. A purse contains shillings and sovereigns. Add a shilling, and then there are twice as many shillings as sovereigns. Add a sovereign to the original contents of the purse, and then there would be more shillings than sovereigns by 2. Find the number of shillings and sovereigns in the purse originally.

Suppose that there were originally in the purse x sovereigns and y shillings.

When a shilling is added there are x sovereigns and $y+1$ shillings, and the first given condition is

$$y+1 = 2x \quad . \quad . \quad . \quad . \quad (1)$$

When a sovereign is added there are $x+1$ sovereigns and y shillings, and the second given condition is

$$y = x+1+2 \quad . \quad . \quad . \quad (2)$$

(1)−(2) gives

$$1 = x-3, \text{ or } x = 4;$$

$$\text{then} \quad y = x+3 = 7,$$

or there were originally 4 sovereigns and 7 shillings in the purse.

233. 2. A certain number, consisting of two digits, is equal to six times the sum of the digits, and if 117 be subtracted from three times the number, the digits are reversed. Find the number.

Let x be the digit in the place of tens,

y ,, ,, units,

so that $10x+y$ is the number,

and $10y+x$ is the number when the digits are reversed.

$$\therefore\ 10x+y = 6(x+y) \quad . \quad . \quad . \quad (1)$$

$$3(10x+y)-117 = 10y+x \quad . \quad . \quad . \quad (2)$$

$$\text{Hence} \quad 4x = 5y,$$

$$29x-7y = 117.$$

$$\therefore\ x = 5,$$

$$y = 4,$$

and the number is 54.

234. 3. A fraction becomes equal to 2 when 7 is added to its numerator, and equal to 1 when 1 is subtracted from its denominator. Find the fraction.

Let $\dfrac{x}{y}$ be the fraction.

Then the conditions of the question give

$$\frac{x+7}{y} = 2 \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{x}{y-1} = 1 \quad . \quad . \quad . \quad . \quad (2)$$

$$\text{or} \quad 2y-x = 7 \quad . \quad . \quad . \quad . \quad (1)$$

$$y-x = 1 \quad . \quad . \quad . \quad . \quad (2)$$

$$\therefore\ (1)-(2) \text{ gives} \qquad y = 6,$$

$$\therefore\ x = y-1 = 5,$$

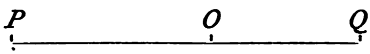
and the fraction is $\frac{5}{6}$.

The following is a similar problem:

4. A certain fraction becomes equal to 1 when 1 is added to the numerator, and equal to $\frac{1}{2}$ when 4 is added to the denominator. What is the fraction?

Ans. The fraction is $\frac{5}{6}$.

235. 5. Two men, A and B, run at uniform paces from one station to another 4000 feet off. A starts 30 seconds after B, and arrives at the second station 10 seconds before B. Where does A pass B?



Let P be the station from which the men start, O the place where one passes the other, Q the station to which they run.

Let A run x feet in a second,
B run y „ „

$$\text{Then A runs over } PO \text{ in } \frac{PO}{x} \text{ seconds,}$$

$$\text{over } OQ \text{ in } \frac{QO}{x} \quad „$$

$$\text{B runs over } PO \text{ in } \frac{PO}{y} \quad „$$

$$\text{over } OQ \text{ in } \frac{QO}{y} \quad „$$

Hence, by the terms of the question, since A takes 30 seconds less to run over PO than B does, and 10 seconds less to run over QO,

$$\frac{PO}{y} - \frac{PO}{x} = 30, \quad . \quad . \quad . \quad (1)$$

$$\frac{QO}{y} - \frac{QO}{x} = 10. \quad . \quad . \quad . \quad (2)$$

Since $PO+OQ=4000$ feet, (1)+(2) gives

$$\frac{4000}{y}-\frac{4000}{x}=40,$$

$$\text{or} \qquad \frac{1}{y}-\frac{1}{x}=\frac{1}{100}.$$

$$\therefore \text{ from (1) } PO\left(\frac{1}{y}-\frac{1}{x}\right) \text{ or } \frac{PO}{100}=30;$$

$$PO=3000 \text{ feet},$$

whence O is known to be 3000 feet from the starting-place.

Though the place where one man runs past the other is thus determined, the given conditions do not suffice to determine x and y, or the speed at which the men run. It will be found that their paces may be arranged in various ways to fulfil the terms of the question. All that is known is that $\frac{1}{x}$, the number of seconds in which A runs a foot, is less by $\frac{1}{100}$ than $\frac{1}{y}$, the number of seconds in which B runs a foot, or that A runs 100 feet in less time by a second than the time which B takes to run that distance.

236. 6. A person walks from A to B, a distance of $9\frac{1}{4}$ miles in 2 hours and 52 minutes, and returns in 2 hours and 44 minutes. His rates of walking up hill, down hill, and on the level being 3, $3\frac{3}{4}$, and $3\frac{1}{4}$ miles an hour respectively. Find the length of level ground between A and B.

In going from A to B let there be

$$\begin{cases} x \text{ miles of ascent, requiring } \dfrac{x}{3} \text{ hours, or } 20x \text{ minutes,} \\ y \quad ,, \quad \text{level,} \quad ,, \quad \dfrac{y}{3\frac{1}{4}} \quad ,, \quad \text{or } \dfrac{240y}{13} \quad ,, \\ z \quad ,, \quad \text{descent,} \quad ,, \quad \dfrac{z}{3\frac{3}{4}} \quad ,, \quad \text{or } 16z \quad ,, \end{cases}$$

Hence, in going from B to A there are

$$\begin{cases} z \text{ miles of ascent, requiring } 20z \text{ minutes,} \\ y \quad ,, \quad \text{level,} \quad ,, \quad \dfrac{240y}{13} \quad ,, \\ x \quad ,, \quad \text{descent,} \quad ,, \quad 16x \quad ,, \end{cases}$$

We now have the equations

$$20x+\frac{240y}{13}+16z=172 \quad . \quad . \quad . \quad (1)$$

$$20z+\frac{240y}{13}+16x=164 \quad . \quad . \quad . \quad (2)$$

$$\text{while } x+y+z=9\tfrac{1}{4} \quad . \quad . \quad . \quad (3)$$

From these $y=3\frac{1}{4}$,

or there are $3\frac{1}{4}$ miles of level ground.

237. 7. At a contested election there are two members to be returned, and three candidates A, B, C. A obtains 1056 votes, B 987, and C 933. Now 85 voted for B and C, 744 for B only, 98 for C only. How many voted for C and A, how many for A and B, how many for A only?

Each elector has six ways of voting, viz.

(1) for A only, which way is adopted by		x electors, suppose,		
(2) „ B	„	744	„	
(3) „ C	„	98	„	
(4) „ B and C	„	85	„	
(5) „ A and C	„	y	„	suppose,
(6) „ A and B	„	z	„	

Hence the votes polled by the several candidates give us the equations,

$$\begin{aligned} 1056 &= x+y+z, \quad . \quad . \quad (1) \\ 987 &= 744+85+z, \quad . \quad . \quad (2) \\ 933 &= 98+85+y. \quad . \quad . \quad (3) \end{aligned}$$

Then equation (3) gives $y=750$,

„ (2) „ $z=158$,

„ (1) „ $x=148$.

238. *Problems for Exercise.*

8. A farmer sells to one person 9 horses and 7 cows for 300*l.*, and to another at the same prices 6 horses and 13 cows for the same sum. What were the prices?

Ans. The price of a horse is 24*l.* and of a cow 12*l.*

9. A bill of 25 guineas was paid with crowns and half-guineas, and twice the number of half-guineas exceeded three times the number of crowns by 17. How many were there of each? *Ans.* There were 21 crowns, 40 half-guineas.

10. Find the fraction which if 1 be added to its numerator becomes equal to $\frac{1}{3}$, and if 1 be added to its denominator becomes equal to $\frac{1}{4}$. *Ans.* $\frac{4}{15}$.

11. Find a number of two digits to which if the number formed by reversing the digits be added, the sum will be 121, while if this latter number be subtracted, the remainder is 9. *Ans.* 65.

12. The cost of 6 barrels of beer and 10 of porter is 51*l.*, of 3 barrels of beer and 7 of porter is 32*l.* 2*s.* How much beer can be bought for 30*l.*? *Ans.* 10 barrels.

13. Two numbers are such that if one were increased by 18 it would be double the other, and if the second were diminished by 11 it would be one-third of the former. Find the numbers. *Ans.* 12 and 15 were the numbers.

14. If $ax+b$ is 30 when x is 2, and 90 when x is 5, what is its value when x is 3·5, and what value of x makes it zero?

Ans. Its value when $x = 3·5$ is 60, and $x = ·5$ makes it zero.

15. The sum of the ages of a father and son is half what it will be in 25 years. The difference is one-third what the sum will be in 20 years. Find their respective ages?

Ans. The father is 40 years of age, the son 10 years.

16. A and B began to play. After a certain number of games A had won half as much as he had at first, and then

found that if he had 15 shillings more he would have three times as much as B had left. But B afterwards won 10 shillings, and he had then twice as much as A. What had each at first?

Ans. A had 14 shillings, B had 19 shillings.

17. The value of the contents of a bag containing only crowns and half-sovereigns is 628*l.* 10*s.* The value of a bag of silver of the same weight would be 49*l.* 10*s.* Find the number of crowns and of half-sovereigns in the bag.

Note.—623 half-sovereigns weigh as much as 88 crowns.

Ans. 1246 half-sovereigns and 22 crowns.

18. Seventeen gold coins, all of equal value, and as many silver coins, all of equal value, are placed in a row at random. A is to have one half of the row, B the other half. A's share is found to include seven gold coins, and the value of it is 6*l.* The value of B's share is 6*l.* 15*s.* Find the value of each gold and silver coin.

Ans. The gold coins are half-sovereigns, the silver coins are crowns.

19. The road from A to D passes through B and C successively. The distance between A and B is 6 miles greater than that between C and D, the distance between A and C is $\frac{1}{16}$ of a mile short of being half as great again as that between B and D, and the point half-way from A to D is between B and C half a mile from B. Determine the distances between A and B, B and C, C and D.

Ans. From A to B $11\frac{1}{8}$ miles, B to C 7 miles, C to D $5\frac{1}{8}$ miles.

20. Fifteen octavos and twelve duodecimo volumes are arranged on a table, occupying the whole of it. After six of the octavos and four of the duodecimos are removed only $\frac{5}{8}$ of the table is occupied. How many duodecimos only, or octavos only, might be arranged similarly on the table?

Ans. 24 octavos or 32 duodecimos.

21. A merchant made a mixture of wine at 28*s.* a gallon with brandy at 42*s.* a gallon, and found that by selling the mixture at 38*s.* a gallon he gained 15 per cent. on the price of the wine and 20 per cent. on the price of the brandy. In what ratio were the wine and brandy mixed together?

Ans. 29 parts of brandy are united with 62 parts of wine.

22. A laid out money in buying shares of two companies. If he had sold out 6 months later he would have lost (by sale) 35*l.* less than half of his outlay in the shares of the first company, $11\frac{2}{3}$*l.* more than $\frac{1}{3}$ of his outlay in those of the second, and on the whole would have received 10*l.* less than $\frac{5}{8}$ of his original outlay. When he did sell out he lost (by sale) 10*l.* less than $\frac{1}{5}$ of his outlay in the first company, and gained 14*l.* less than $\frac{1}{10}$ of his outlay in the second, so that his loss on the whole together with $\frac{1}{20}$ of his original outlay made 94*l.* Find the sums laid out in the shares of each company.

Ans. 500*l.* was laid out in the first company, 700*l.* in the second.

23. P started at 9 o'clock from his house, and rode to a village at such a rate that if it had been 2 miles further he would have taken just an hour to get to it. He returned without delay at the same rate, and went on to a station 5 miles beyond his house, where he met Q who had just arrived, having started at half-past ten, and made a journey $1\frac{1}{2}$ mile short of being half as long again as P's, at a rate 8 miles an hour short of being six times that of P. Find the rate (in miles per hour) at which each travelled?

Ans. P travelled 4 miles an hour, and Q travelled 16 miles an hour.

CHAPTER VII.

QUADRATIC EQUATIONS, AND PROBLEMS PRODUCING THEM.

239. *Def.*—When an equation, cleared if necessary of fractions or roots affecting the unknown quantity, contains the second power or square of that unknown quantity, it is called a quadratic equation, or an equation of the second degree.

Quadratic equations, therefore, may be of two kinds: the first, those which contain the square of the unknown quantity without the first power; the second, those which contain the square and also the first power of the unknown quantity. The first kind are sometimes named Pure Quadratics, the second Adfected Quadratics.

Thus $8x^2 + \frac{3x^2 - 1}{7} = 5$ is a pure quadratic.

$8x^2 + \frac{3x - 1}{7} = 5$ is an adfected quadratic.

240. The solution of pure quadratics requires no more knowledge than the reader is expected to possess already, for if the methods of solution used on simple equations be applied to them, they lead to the result of bringing the square of the unknown quantity exhibited as a known value, and then the quantity itself is also known.

$$\text{Ex. } x^2 - \frac{x^2 - 1}{3} = 3.$$

If the equation be multiplied throughout by 3,

$$3x^2 - x^2 + 1 = 9,$$

and by transposition,

$$2x^2 = 8,$$
$$x^2 = 4,$$
$$\text{then } x = \pm 2.$$

The sign $\pm$ means that either the positive or negative sign may be used.

241. It is sometimes not at once seen why in such a case as that just before us, $x^2 = 4$, when the square root is extracted, the double sign is not affixed on both sides, and $\pm x = \pm 2$ given as the result. Now let it be considered what fact is to be algebraically expressed: it is that x is either 2 or -2. If double signs be used on both sides this fact will not be expressed. For if they are used with the understanding that the upper signs on each side are to be taken together, or the lower signs taken together, the statement then amounts to no more than $x = 2$, or $-x = -2$, which latter is included in the former. If we are to understand that the upper sign on one side is to correspond with the lower sign on the other, then the fact stated is that $x = -2$, or $-x = 2$, which is the same thing. In either case, therefore, only one half of the whole truth is expressed. But if we write $x = \pm 2$, all that is intended is then stated, namely, that x as to value is 2, and that it may accept either the positive or negative sign.

242. The following are examples of pure quadratics:

Ex. 1. $$\frac{x^2}{16} - \frac{x^2 - 3}{5} = \frac{1}{20}.$$

If the equation be multiplied throughout by 80, the denominators of the fractions will be removed, and

$$5x^2 - 16(x^2 - 3) = 4,$$
$$11x^2 = 44,$$
$$x^2 = 4,$$
$$x = \pm 2.$$

243. Ex. 2. $\sqrt{x^2-5}+\sqrt{x^2+7}=2x.$

By transposition

$$\sqrt{x^2+7}=2x-\sqrt{x^2-5}.$$

If each side of the equation be squared,

$$x^2+7=4x^2-4x\sqrt{x^2-5}+x^2-5.$$

$$\therefore\ 4x\sqrt{x^2-5}=4x^2-12,$$

$$x\sqrt{x^2-5}=x^2-3.$$

By squaring both sides

$$x^4-5x^2=x^4-6x^2+9,$$

$$x^2=9,$$

$$x=\pm 3.$$

By substituting these values in the equation it will be seen that as the positive or negative root is adopted, the square roots must be accepted accordingly, with the positive or negative sign, either of which they are open to receive.

244. The following equations prove to be pure quadratics:

Ex. 3. $\dfrac{1}{\sqrt{a+x}-\sqrt{a}}+\dfrac{1}{\sqrt{a-x}+\sqrt{a}}=\dfrac{\sqrt{a}}{x}.$

$$\frac{1}{\sqrt{a+x}-\sqrt{a}}=\frac{\sqrt{a+x}+\sqrt{a}}{(\sqrt{a+x}-\sqrt{a})(\sqrt{a+x}+\sqrt{a})}$$

$$=\frac{\sqrt{a+x}+\sqrt{a}}{a+x-a}$$

$$=\frac{\sqrt{a+x}+\sqrt{a}}{x},$$

and $\dfrac{1}{\sqrt{a-x}+\sqrt{a}}=\dfrac{\sqrt{a}-\sqrt{a-x}}{(\sqrt{a}+\sqrt{a-x})(\sqrt{a}-\sqrt{a-x})}$

$$=\frac{\sqrt{a}-\sqrt{a-x}}{x}.$$

Hence the equation takes the form:

$$\frac{\sqrt{a+x}+\sqrt{a}+\sqrt{a}-\sqrt{a-x}}{x}=\frac{\sqrt{a}}{x},$$

$$\text{or}\quad \sqrt{a+x}-\sqrt{a-x}+2\sqrt{a}=\sqrt{a},$$

$$\text{or}\quad \sqrt{a+x}-\sqrt{a-x}=-\sqrt{a}.$$

If each side be now squared,

$$2a-2\sqrt{a^2-x^2}=a,$$
$$2\sqrt{a^2-x^2}=a,$$
$$4(a^2-x^2)=a^2,$$
$$4x^2=3a^2,$$
$$x=\pm\frac{\sqrt{3}}{2}a.$$

245. Ex. 4. $\dfrac{\sqrt{b+x}-\sqrt{a-x}}{\sqrt{a+x}-\sqrt{b-x}}=\dfrac{2x-a+b}{2x+a-b}.$

Since $\dfrac{2x-a+b}{2x+a-b}=\dfrac{b+x-(a-x)}{a+x-(b-x)},$

$$=\frac{(\sqrt{b+x}-\sqrt{a-x})(\sqrt{b+x}+\sqrt{a-x})}{(\sqrt{a+x}-\sqrt{b-x})(\sqrt{a+x}+\sqrt{b-x})},$$

we have first $\sqrt{b+x}-\sqrt{a-x}=0,$

$$\therefore\ b+x=a-x,$$
$$x=\tfrac{1}{2}(a-b);$$

or if x has another value,

$$1=\frac{\sqrt{b+x}+\sqrt{a-x}}{\sqrt{a+x}+\sqrt{b-x}},$$

$$\sqrt{a+x}+\sqrt{b-x}=\sqrt{b+x}+\sqrt{a-x},$$

$$\sqrt{a+x}-\sqrt{a-x}=\sqrt{b+x}-\sqrt{b-x}.$$

If each side be squared,

$$a - \sqrt{a^2 - x^2} = b - \sqrt{b^2 - x^2},$$
$$\sqrt{a^2 - x^2} - \sqrt{b^2 - x^2} = a - b.$$
$$\therefore\ a^2 + b^2 - 2x^2 - 2\sqrt{(a^2 - x^2)(b^2 - x^2)} = a^2 + b^2 - 2ab,$$
$$\sqrt{(a^2 - x^2)(b^2 - x^2)} = ab - x^2,$$
$$a^2b^2 - (a^2 + b^2)x^2 + x^4 = a^2b^2 - 2abx^2 + x^4,$$
$$(a^2 - 2ab + b^2)x^2 = 0.$$
$$\therefore\ x = 0.$$

Thus pure quadratics require for their solution no knowledge of processes beyond those already applied to simple equations.

246. Adfected quadratics depend for their solution on the form of a squared binomial,

$$(x + a)^2 = x^2 + 2ax + a^2.$$

Hence, when a quantity, $x^2 + 2ax$, is presented, it will become a complete square if a^2, i.e. the square of half the coefficient of x, be added to it. By help of our liberty of adding the same quantity to both members of an equation (134) adfected quadratics are thus brought under solution.

If, for instance, $\quad x^2 + 8x = 9,$

it will be observed, after the preceding remark, that $x^2 + 8x$ will become a square if 16 be added to it. If then 16 be added to each side of the equation,

$$x^2 + 8x + 16 = 9 + 16 = 25.$$

If the square root of each side be now taken,

$$x + 4 = \pm 5 \text{ (241)}.$$
$$\therefore\ x = \pm 5 - 4 = 1, \text{ or } -9.$$

Two values of the unknown quantity are therefore obtained, and it will be found on substitution that $x^2 + 8x$ takes the same value 9 whether 1 or -9 is inserted as x.

247. It will be observed that

$$x^2+8x-9 = (x-1)(x+9),$$

and the equation may be written

$$(x-1)(x+9) = 0.$$

The first member, being the product of two factors, can only be zero by one of the factors being zero (67).

Hence we must either have

$$x-1 = 0,\ x = 1,$$

$$\text{or,}\quad x+9 = 0,\ x = -9,$$

and these values of x agree with the roots of the equation already obtained.

248. The method of solving an adfected quadratic may therefore be described in the form of a rule. The equation, it is supposed, is presented, clear of fractions and roots affecting the unknown quantity, with terms in x collected in the first member, in the form

$$ax^2+bx = c.$$

1. Divide throughout by the coefficient of x^2, so that x^2 stands as the first term.

2. Add to each side the square of half the coefficient of x, 'completing the square,' as this process is called.

3. Extract the square root of each member and the equation will be reduced to two simple equations, and the values of x will be obtained after a transposition.

249. Ex. 1. $2x^2-7x = -3$.

Following the order of the rule we have,

1. $x^2-\frac{7}{2}x = -\frac{3}{2}$.

2. $x^2-\frac{7}{2}x+(\frac{7}{4})^2 = (\frac{7}{4})^2-\frac{3}{2} = \frac{25}{16}$.

3. $x-\frac{7}{4} = \pm\frac{5}{4}$.

$\therefore$ (1) $x-\frac{7}{4} = \frac{5}{4}$,

$$x = \tfrac{12}{4} = 3.$$

(2) $x-\frac{7}{4} = -\frac{5}{4}$,

$$x = \tfrac{2}{4} = \tfrac{1}{2}.$$

Ex. 2. $-4x^2+7x = -2.$

1. $x^2-\frac{7}{4}x = \frac{2}{4}.$
2. $x^2-\frac{7}{4}x+(\frac{7}{8})^2 = (\frac{7}{8})^2+\frac{2}{4} = \frac{81}{64}.$
3. $x-\frac{7}{8} = \pm\frac{9}{8}.$

$\therefore$ (1) $x = \frac{7}{8}+\frac{9}{8} = 2,$

(2) $x = \frac{7}{8}-\frac{9}{8} = -\frac{1}{4}.$

250. Ex. 3. $x^2+169-\dfrac{156x}{5}+\dfrac{36x^2}{25} = 74.$

Science Examination 1866.

To clear the equation of fractions let it be multiplied throughout by 25.

$$\therefore\ 25x^2+4225-780x+36x^2 = 1850,$$

$$\text{or}\qquad 61x^2-780x = -2375,$$

$$x^2-\frac{780}{61}x+\left(\frac{390}{61}\right)^2 = -\frac{2375}{61}+\left(\frac{390}{61}\right)^2$$

$$= \frac{7225}{(61)^2},$$

$$\therefore\ x-\frac{390}{61} = \pm\frac{85}{61}.$$

$$x = \frac{390\pm85}{61} = \frac{475}{61} \text{ or } 5.$$

251. Ex. 4. $4x^2-12x+9 = 0.$

In pursuance of this rule the equation is first altered into the form

$$x^2-3x = -\tfrac{9}{4},$$

and the quantity to be added to complete the square is $(-\frac{3}{2})^2$ or $\frac{9}{4}$,

$$\therefore\ x^2-3x+\tfrac{9}{4} = \tfrac{9}{4}-\tfrac{9}{4} = 0,$$

$$\therefore\ x-\tfrac{3}{2} = 0,$$

$$x = \tfrac{3}{2}.$$

In this instance, therefore, only one value of x results, the two factors into which, as we have seen, a quadratic is reducible (247), merging in this instance into $(x-\frac{3}{2})^2$.

252. Ex. 5. $2x^2-3x+9=0.$

$$x^2-\frac{3}{2}x=-\frac{9}{2}.$$

The quantity to be added to complete the square being

$$(-\tfrac{3}{2})^2 \text{ or } \tfrac{9}{4},$$

$$x^2-\frac{3}{2}x+\frac{9}{4}=\frac{9}{4}-\frac{9}{2}=-\frac{9}{4}.$$

Here while the square root of the first member can be extracted, the square root of the latter member cannot be extracted in any numerical form (33). Indeed the result at which we have already arrived expresses an impossibility if x has any numerical value, since $(x-\frac{3}{2})^2$, a quantity which can never be negative, is represented to be equal to the negative quantity $-\frac{9}{4}$. The inference to be drawn is that the proposed equation cannot hold true with any value of x that can be numerically expressed. Such an equation is called an equation with impossible roots.

253. Quantities which admit of numerical representation either exactly, or as nearly to exactness as we please by continuation of interminable decimals, are called real quantities, whereas those which can never be numerically expressed, such as the square root of a negative quantity, are called unreal, impossible, or imaginary quantities.

In this book, which has for its object instruction in Algebra as a method of calculation, no consideration will be given to these unreal quantities.

254. Thus while a simple equation, wherein x stands in connection with numerical quantities, never fails to lead to the assigning of a single definite value to x when the proper methods are used for the solution of the equation, it appears that we have not the same security with a quadratic. Two different roots, we have seen, in some instances result, one root in another case, and no numerical root in a third.

255. It is convenient to have a means of deciding under which of these cases a quadratic equation falls before its

solution is commenced, and this power is supplied by the following test.

$$\text{Let} \quad ax^2+bx+c=0$$

be a quadratic equation in its general form, a, b, c being any given quantities. If the rule of solution be followed,

$$x^2+\frac{b}{a}x=-\frac{c}{a},$$

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a},$$

$$=\frac{b^2-4ac}{4a^2}.$$

Now (1) if b^2 is greater than $4ac$, then the fraction

$$\frac{b^2-4ac}{4a^2}$$

is a positive quantity whose square root can be extracted either exactly or with as near an approach to exactness as we please. If then d be the numerical value of this square root,

$$x+\frac{b}{2a}=\pm d,$$

$$x=-\frac{b}{2a}\pm d,$$

and two different real values for x are obtained.

(2) If $b^2=4ac$,

$$\text{then} \quad x^2+\frac{bx}{a}+\frac{b^2}{4a^2}=0,$$

$$\therefore x+\frac{b}{2a}=0,$$

$$\text{or} \quad x=-\frac{b}{2a},$$

and only one value of x arises.

(3) If b^2 is less than $4ac$, $\frac{b^2-4ac}{4a^2}$ is negative:

Then $\left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$, a negative quantity,

or a quantity which can never be negative appears to equal a negative quantity. This is then the case when the equation cannot be satisfied by any real value of x, because such a supposition pursued to its consequences leads to an impossibility.

Thus when a quadratic equation

$$ax^2+bx+c=0$$

is presented, the sign of b^2-4ac decides whether it has two, one, or no real value of x satisfying it.

If $b^2>4ac$ there are two different real roots.

If $b^2=4ac$ there is one real value of x arising, or the two roots which a quadratic generally has are equal.

If $b^2<4ac$, there is no possible root.

Ex. $3297x^2-54x+123=0$.

Since $(54)^2<4\times 123\times 3297$,

this equation cannot be satisfied by any real value of x.

256. Determine in the following cases, before the solution is attempted, whether the roots will be real or not.

1. $397x^2+824x+25=0$.
2. $25x^2-70x+49=0$.
3. $112x^2-86x+3=0$.

257. If $ax^2+bx+c=0$ has real roots, the expression ax^2+bx+c can be resolved into two real factors, each containing the first power of x.

$$\text{For } ax^2+bx+c = a\left(x^2+\frac{bx}{a}+\frac{c}{a}\right),$$

$$= a\left\{x^2+\frac{bx}{a}+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}+\frac{c}{a}\right\},$$

$$= a\left\{\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right\}.$$

Now by the fact of $ax^2+bx+c=0$ having real roots, $\frac{b^2-4ac}{4a^2}$ is some positive quantity. Let it be called d^2.

Then $ax^2+bx+c=a\left\{\left(x+\frac{b}{2a}\right)^2-d^2\right\}$

$$=a\left\{x+\frac{b}{2a}+d\right\}\left\{x+\frac{b}{2a}-d\right\},$$

and is thus resolved into factors.

It will be observed that

$$-\frac{b}{2a}-d \text{ and } -\frac{b}{2a}+d$$

are the roots of the equation

$$x^2+\frac{bx}{a}+\frac{c}{a}=0.$$

258. When x is open to admit all values whatever, positive or negative, the expression $ax+b$ receives accordingly all values by the variation of x, a and b having certain assigned invariable values. This will not always be the case with the expression ax^2+bx+c.

For $ax^2+bx+c=a\left\{x^2+\frac{b}{a}x+\frac{c}{a}\right\}$,

$$=a\left\{x^2+\frac{b}{c}x+\frac{b^2}{4a^2}+\frac{c}{a}-\frac{b^2}{4a^2}\right\},$$

$$=a\left\{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right\}.$$

Now, if $b^2<4ac$ the quantity within the brackets is always positive, and can range in value from $\frac{4ac-b^2}{4a^2}$ upwards. Hence the expression ax^2+bx+c always has in this case the same sign as a, and can range in value from $\frac{4ac-b^2}{4a}$ as its limit.

If $b^2>4ac$, then ax^2+bx+c can become zero.

If $\dfrac{4ac-b^2}{4a^2}$, as it is necessarily a negative quantity, is represented by $-n^2$,

$$ax^2+bx+c = a\left\{\left(x+\frac{b}{2a}\right)^2-n^2\right\},$$

and is zero when $x+\dfrac{b}{2a} = \pm n$.

$$\text{Let } -\frac{b}{2a}+n = h,$$

$$\text{and } -\frac{b}{2a}-n = k.$$

$$\therefore\ ax^2+bx+c = a(x-h)(x-k).$$

As long as x is greater than both h and k or less than both h and k, the expression has the same sign with a, and in numerical magnitude can increase without limit.

When x is between h and k, the expression has a contrary sign to a.

Ex. 1.
$$\begin{aligned} 3x^2+3x+1 &= 3\ (x^2+x+\tfrac{1}{3}), \\ &= 3(x^2+x+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{12}), \\ &= 3\{(x+\tfrac{1}{2})^2+\tfrac{1}{12}\}. \end{aligned}$$

This expression is always positive whatever be the value of x, and its least value is $\frac{1}{4}$, when $x = -\frac{1}{2}$, because of the two terms of which it consists, that one which is open to variation is then made to have its least value.

Ex. 2.
$$\begin{aligned} 3x^2-15x+18 &= 3(x^2-5x+6), \\ &= 3(x^2-5x+\tfrac{25}{4}-\tfrac{1}{4}), \\ &= 3\{(x-\tfrac{5}{2})^2-\tfrac{1}{4}\}, \\ &= 3(x-2)(x-3). \end{aligned}$$

Hence as long as x is greater than 3, the expression is positive and can be made to take any numerical value.

When x is between 2 and 3 the expression is negative.

When x is less than 2 the expression is positive and can be made to take any numerical value.

Ex. 3. $4x-x^2-3$ can never exceed unity whatever value x assumes.

$$\begin{aligned}\text{For } 4x-x^2-3 &= 1-x^2+4x-4,\\ &= 1-(x-2)^2,\end{aligned}$$

and therefore is unity lessened by a quantity which must always be positive or zero.

The expression is equal to unity when $x = 2$.

259. When two roots of a quadratic have been found, whether they be two different roots as in the first case of the three above-mentioned (255), or two equal roots as in the second case, no more are to be obtained, and the solution is complete. For were it possible that the quadratic could have more than two different roots, let these be p, q, r. Then the characteristic of a root being that it satisfies the equation (133)

$$\text{we have } \left.\begin{aligned} ap^2+bp+c &= 0\\ aq^2+bq+c &= 0\\ ar^2+br+c &= 0\end{aligned}\right\}\quad \begin{aligned}(1)\\(2)\\(3)\end{aligned}$$

Now (1)—(2) gives

$$a(p^2-q^2)+b(p-q) = 0.$$

Since p is unequal to q this equation may be divided by $p-q$, and then

$$a(p+q)+b = 0.$$

So (1)—(3) gives

$$a(p+r)+b = 0,$$

$\therefore\ r = q$, or the three roots are not different.

Also if two roots be equal there cannot be a third, for if p be one of the equal roots, since in this case the equation as it is presented requires no completion of the square, and $b^2 = 4ac$,

$$ap^2+bp+c = 0$$

$$\text{gives } ap^2 \pm 2\sqrt{ac}.p+c = 0,$$

$$\therefore\ p\sqrt{a} \pm \sqrt{c} = 0.$$

If then r were another root different from p,

$$r\sqrt{a} \pm \sqrt{c} = 0$$

would be true for similar reasons,

$$\therefore\ p = r,$$

or p and r are not different.

260. $$x^2 - {\cdot}075x + 3{\cdot}45 = 0.$$

Science Examination, 1865.

$$x^2 - {\cdot}075x = -3{\cdot}45$$
$$x^2 - {\cdot}075x + ({\cdot}0375)^2 = ({\cdot}0375)^2 - 3{\cdot}45$$
$$= \text{a negative quantity,}$$

$\therefore\ x - {\cdot}0375 =$ the square root of a negative quantity.

$\therefore\ x$ can have no real value to satisfy this equation.

This is an example of the appearance of what are termed impossible roots. Their existence may be seen before the solution of the equation is attempted by the test of (255).

261. *Examples for Practice.*

1. $x^2 - 34 = \frac{x}{3}$. *Ans.* $x = 6$, or $-\frac{17}{3}$.

2. $2x^2 + x = 15$. *Ans.* $x = \frac{5}{2}$, or -3.

3. $ab(x^2 + 1) = (a^2 + b^2)x$. *Ans.* $x = \frac{a}{b}$, or $\frac{b}{a}$.

4. $(x-3)(x-2) + (x-3)(x-1) + (x-1)(x-2) = 2$. *Ans.* $x = 3$, or 1.

5. $x^2 - 78 = \frac{x}{3}$. *Ans.* $x = 9$, or $-\frac{26}{3}$.

6. $(a+b)(ax+b)(a-bx) = (a^2x - b^2)(a+bx)$. *Ans.* $x = 1$, or $-\frac{a+2b}{b+2a}$.

7. $\frac{x^2}{3} - \frac{x}{2} = 9.$ *Ans.* $x = 6$, or $-\frac{9}{2}$.

8. $3x^2 - 53x + 34 = 0.$ *Ans.* $x = 17$, or $\frac{2}{3}$.

9. $(x-3)(x-5) = 35.$ *Ans.* $x = 10$, or -2.

10. $(x+2)^2 = 4(x+5).$ *Ans.* $x = \pm 4$.

11. $x^2 + \frac{x}{2} = \frac{1}{2}.$ *Ans.* $x = \frac{1}{2}$, or -1.

12. $\frac{x^2}{2} - \frac{x}{3} = \frac{5}{8}.$ *Ans.* $x = \frac{3}{2}$, or $-\frac{5}{6}$.

13. $3x^2 - 5x - 12 = 0.$ *Ans.* $x = 3$, or $-\frac{4}{3}$.

14. $6x^2 = 5x + 6.$ *Ans.* $x = \frac{3}{2}$, or $-\frac{2}{3}$.

15. $x = \frac{5}{3} + \frac{1}{12}x^2.$ *Ans.* $x = 10$, or 2.

16. $\frac{x}{a} + \frac{a}{x} = \frac{x}{b} + \frac{b}{x}.$ *Ans.* $x = \pm\sqrt{ab}.$

17. $\frac{m}{x} + nx = mx + \frac{n}{x}.$ *Ans.* $x = \pm 1$.

18. $x^2 - (a^2+b^2)x + (a^2-b^2)ab = 0.$
Ans. $x = \frac{1}{2}(a^2+b^2) \pm \frac{1}{2}\sqrt{a^4 - 4a^3b + 2a^2b^2 + 4ab^3 + b^4}.$

19. $x - \frac{x^3-8}{x^2+5} = 2.$ *Ans.* $x = 2$, or $\frac{1}{2}$.

20. $5x + \frac{4x+3}{3x-2} = \frac{10x^2-7}{2x-1} + 9.$ *Ans.* $x = 1$, or $\frac{35}{61}$.

21. $\frac{1}{x-2} - \frac{2}{x+2} = \frac{3}{5}.$ *Ans.* $x = 3$, or $-\frac{14}{3}$.

22. $\sqrt{3x-2}\sqrt{2x-3} = 12.$ *Ans.* $x = 6$, or $-\frac{23}{6}$.

23. $\sqrt{2x+4} - \sqrt{\frac{x}{2}+6} = 1.$ (243) *Ans.* $x = 6$, or $-\frac{10}{9}$.

24. $\sqrt{x+2} + \sqrt{2x+2} = x.$ *Ans.* $x = 7$, or -1.

25. $(2x-3)(x+4) = 63.$ *Ans.* $x = 5$, or $-\frac{15}{2}$.

26. $x+\frac{1}{x} = 3.$ *Ans.* $x = \frac{3 \pm \sqrt{5}}{2}$.

27. $15x-\frac{11}{x} = \frac{13x}{5}$. (240) *Ans.* $x = \pm\sqrt{\frac{55}{62}}$.

28. $\sqrt{x+3}+\sqrt{x+6} = 3\sqrt{x}.$ *Ans.* $x = \frac{9 \pm 2\sqrt{19}}{5}$.

29. $\frac{x+2}{x-1}-\frac{x-1}{x+2} = \frac{7}{12}.$ *Ans.* $x = 10$, or $-\frac{5}{7}$.

30. $\frac{3}{x+1}-8 = \frac{3}{x-1}.$ *Ans.* $x = \pm\frac{1}{2}$.

31. $\sqrt{82+x}-\sqrt{82-x} = 2$. (243) *Ans.* $x = \pm 18$.

32. $\sqrt{50+x}-\sqrt{50-x} = 2.$ *Ans.* $x = \pm 14$.

33. $2\sqrt{x^2-4x}+4x = 1.$ *Ans.* $x = -\frac{1}{2}$, or $-\frac{1}{6}$.

34. $\frac{x-4}{x+4} = \frac{x+3}{x-3}+\frac{7x}{6}.$ *Ans.* $x = 0$, or -1.

35. $\sqrt{3+x}+\sqrt{3-x} = 2\sqrt{x}$. (243) *Ans.* $x = 0$, or $2\frac{2}{5}$.

36. $\frac{5x}{x-2}-\frac{8}{x+1} = 5+\frac{3(2x+1)}{x^2-4}.$ *Ans.* $x = 7$, or $-\frac{7}{4}$.

37. $\frac{5x-9}{x+3}+\frac{8x+44}{4x-8} = 9.$ *Ans.* $x = 5$, or $-\frac{21}{2}$.

38. $4x^2+x\sqrt{2} = 1.$ *Ans.* $x = \frac{1}{4}\sqrt{2}$, or $-\frac{1}{2}\sqrt{2}$.

39. $\frac{9x^2+1}{3x-2}-3x = \frac{5x-1}{4x-3}+3.$ *Ans.* $x = 1$, or $\frac{23}{27}$.

40. $x-\frac{5}{2}+\frac{1}{x} = 0.$ *Ans.* $x = 2$, or $\frac{1}{2}$.

41. $(x-9\frac{1}{3})^2 = 4(x-10\frac{1}{3})$. (251) *Ans.* $x = 11\frac{1}{3}$.

42. If $\{2a+(n-1)b\}\frac{n}{2}$ is 48 when $a = \frac{1}{2}$ and $b = \frac{1}{3}$, find a positive integral value for n. *Ans.* $n = 16$.

262. Occasionally equations which involve more than the first and second powers of the unknown quantity, and so do not fulfil the definition of a quadratic, happen from their peculiar form to be reducible to quadratics.

The following are instances.

263. Ex. 1. $x^3+8 = (x+2)(5x+8)$.

Each side of the equation is divisible by $x+2$. Hence, if $x = -2$ the equation is satisfied, or -2 is one root.

If x have any other value than -2, then after dividing by $x+2$.

$$\text{we have } x^2-2x+4 = 5x+8,$$
$$x^2-7x+\tfrac{49}{4} = 4+\tfrac{49}{4} = \tfrac{65}{4},$$
$$x = \frac{7\pm\sqrt{65}}{2}.$$

So that the three roots of the equation are

$$-2,\ \frac{7+\sqrt{65}}{2},\ \frac{7-\sqrt{65}}{2}.$$

Ex. 2.
$$\frac{1+x^3}{(1+x)^3} = \tfrac{13}{25}.$$
$$\therefore \frac{1-x+x^2}{(1+x)^2} = \tfrac{13}{25}.$$
$$25(1-x+x^2) = 13(1+2x+x^2);$$
$$12x^2-51x = -12,$$
$$x^2-\tfrac{51}{12}x+(\tfrac{51}{24})^2 = -1+(\tfrac{51}{24})^2$$
$$x = \frac{51\pm45}{24} = \tfrac{96}{24} \text{ or } \tfrac{6}{24},$$
$$= 4 \text{ or } \tfrac{1}{4}.$$

Ex. 3. $(x-2)(x-4)(x-6) = 3(x-2)^2$.

In this equation one root $x = 2$ is found at once by inspection, each member being divisible by $x-2$, and therefore becoming zero when $x = 2$.

If the equation is satisfied by other values of x than 2,

$$(x-4)(x-6) = 3(x-2),$$
$$x^2 - 10x + 24 = 3x - 6,$$
$$x^2 - 13x + (\tfrac{13}{2})^2 = -30 + \tfrac{169}{4} = \tfrac{49}{4},$$
$$x = \frac{13 \pm 7}{2} = 10 \text{ or } 3.$$

$\therefore$ 2, 3, 10 are the values of x which satisfy the proposed equation.

264. The following adfected quadratic happens to admit of being solved without any completion of the square:

$$\frac{x^2 + c^2}{2cx} = \frac{a^2 + 3ab + b^2}{a^2 + ab + b^2}.$$

Hence $$\frac{x^2 + 2cx + c^2}{x^2 - 2cx + c^2} = \frac{a^2 + 3ab + b^2 + a^2 + ab + b^2}{a^2 + 3ab + b^2 - a^2 - ab} \quad (169),$$

$$= \frac{2(a^2 + 2ab + b^2)}{2ab},$$

or $$\left(\frac{x+c}{x-c}\right)^2 = \frac{(a+b)^2}{ab}.$$

$$\frac{x+c}{x-c} = \pm \frac{a+b}{\sqrt{ab}}.$$

Hence $$\frac{x+c+x-c}{x+c-(x-c)} = \frac{a+b \pm \sqrt{ab}}{a+b \mp \sqrt{ab}} \quad (169),$$

$$\frac{x}{c} = \frac{a+b \pm \sqrt{ab}}{a+b \mp \sqrt{ab}},$$

$$x = c \cdot \frac{a+b \pm \sqrt{ab}}{a+b \mp \sqrt{ab}}.$$

265. Although a quadratic equation has been defined to be one which, when reduced to its simplest form, contains only the first and second powers of the unknown quantity, yet many equations which contain higher powers of that quantity admit of being solved like quadratics by an artificial completion of the square, when they contain some

power or expression and the square of the same. This will be observed in the following examples.

Ex. 1. $x^6 - 11x^3 + 24 = 0.$

This is analogous to a quadratic in its containing x^3 and x^6 which is $(x^3)^2$. It can be solved like a quadratic.

$$x^6 - 11x^3 = -24,$$
$$x^6 - 11x^3 + (\tfrac{11}{2})^2 = (\tfrac{11}{2})^2 - 24 = \tfrac{25}{4},$$
$$x^3 - \tfrac{11}{2} = \pm\tfrac{5}{2},$$
$$x^3 = \frac{11 \pm 5}{2} = 8 \text{ or } 3,$$
$$\therefore x = 2 \text{ or } \sqrt[3]{3}.$$

There are other unreal roots also.

266. Ex. 2. $x^2 + 2\sqrt{x^2 - 2x} = 2(x+4).$

$$\therefore x^2 - 2x + 2\sqrt{x^2 - 2x} = 8.$$

It is now observable that the equation contains $\sqrt{x^2 - 2x}$ and also $x^2 - 2x$ which is $(\sqrt{x^2 - 2x})^2$. It can thus be solved like a quadratic.

$$x^2 - 2x + 2\sqrt{x^2 - 2x} + 1 = 9,$$
$$\sqrt{x^2 - 2x} + 1 = \pm 3,$$
$$\sqrt{x^2 - 2x} = -4 \text{ or } 2,$$
$$\therefore x^2 - 2x = 16 \text{ or } 4.$$

We have now brought the equation to two quadratics.

1. $x^2 - 2x = 16,$
$$x^2 - 2x + 1 = 17,$$
$$x = 1 \pm \sqrt{17}.$$

2. $x^2 - 2x = 4,$
$$x^2 - 2x + 1 = 5,$$
$$x = 1 \pm \sqrt{5}.$$

Thus the four roots of the equation are obtained.

It will be seen already that the methods to be employed

with equations like these cannot be taught by any specific rules, but must be drawn from the student's own ingenuity, when he has cultivated it by the study of examples worked for his instruction.

A few examples will accordingly be given here with the steps of the solution exhibited, though the purposes of this book do not allow or require anything like a complete display of the artifices which equations bring into use.

267. $$4x^2+8x+\frac{8}{x}+\frac{4}{x^2} = 37.$$

Since $$\left(x+\frac{1}{x}\right)^2 = x^2+2+\frac{1}{x^2},$$

if 8 be added to each side of the equation,

$$4\left(x+\frac{1}{x}\right)^2+8\left(x+\frac{1}{x}\right) = 45,$$

$$4\left(x+\frac{1}{x}\right)^2+8\left(x+\frac{1}{x}\right)+4 = 49,$$

$$2\left(x+\frac{1}{x}\right)+2 = \pm 7,$$

$$x+\frac{1}{x} = \tfrac{5}{2} \text{ or } -\tfrac{9}{2}.$$

1. $$x+\frac{1}{x} = \tfrac{5}{2}.$$

If 2 be added to and subtracted from each side successively,

$$x+2+\frac{1}{x} = \tfrac{9}{2},$$

and $$x-2+\frac{1}{x} = \tfrac{1}{2},$$

whence $$\sqrt{x}+\frac{1}{\sqrt{x}} = \pm\frac{3}{\sqrt{2}},$$

and $$\sqrt{x}-\frac{1}{\sqrt{x}} = \pm\frac{1}{\sqrt{2}},$$

By addition $$2\sqrt{x} = \pm\frac{4}{\sqrt{2}} \text{ or } \pm\frac{2}{\sqrt{2}},$$
$$4x = 8 \text{ or } 2,$$
$$x = 2 \text{ or } \tfrac{1}{2}.$$

$$2. \quad x+\frac{1}{x} = -\tfrac{9}{2}.$$

Under the same process,
$$\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 = -\tfrac{5}{2},$$
$$\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 = -\tfrac{13}{2},$$
or quantities which can never be negative for any real value of x are equal to negative quantities. Hence this subdivision of the solution gives no real value of x, though it might lead to forms of solution called impossible roots, which are not here considered.

268. $$\frac{x+4}{x-4}-\frac{x-4}{x+4} = \tfrac{8}{3}. \quad . \quad . \quad . \quad \text{(A)}$$

Suppose that the equation to be solved were
$$x-\frac{1}{x} = \tfrac{8}{3}. \quad . \quad . \quad . \quad \text{(B)}$$

Then $$x^2-1 = \tfrac{8}{3}x,$$
$$x^2-\tfrac{8}{3}x+\tfrac{16}{9} = 1+\tfrac{16}{9} = \tfrac{25}{9},$$
$$x = \frac{4\pm5}{3} = 3 \text{ or } -\tfrac{1}{3}.$$

Now this solution of (B) suggests the manner of solving (A), because (A) coincides in form with (B) if the fraction $\frac{x+4}{x-4}$ be regarded as occupying the place of x.

If then the equation (A) be multiplied throughout by $\frac{x+4}{x-4}$, we have
$$\left(\frac{x+4}{x-4}\right)^2-1 = \tfrac{8}{3}\,\frac{x+4}{x-4},$$
$$\left(\frac{x+4}{x-4}\right)^2-\tfrac{8}{3}\cdot\frac{x+4}{x-4} = 1,$$

$$\text{whence} \quad \frac{x+4}{x-4} = 3 \text{ or } -\tfrac{1}{3}.$$

$\therefore$ 1. $x+4 = 3x-12, \quad x = 8,$

2. $x+4 = -\frac{1}{3}(x-4) \; x = -2.$

269. $$x^2+5x+4 = 5\sqrt{x^2+5x+28}.$$

Let 24 be added to each side of the equation.

$$\therefore x^2+5x+28 = 5\sqrt{x^2+5x+28}+24.$$

There is now an equation containing a quantity $\sqrt{x^2+5x+28}$, and the square of the same. It falls therefore under the treatment of a quadratic, the square being completed by the addition of $(\frac{5}{2})^2$.

$$x^2+5x+28-5\sqrt{x^2+5x+28}+(\tfrac{5}{2})^2 = 24+\tfrac{25}{4} = \tfrac{121}{4}.$$

$$\therefore \sqrt{x^2+5x+28} = \frac{5\pm 11}{2} = 8 \text{ or } -3,$$

$$x^2+5x+28 = 64 \text{ or } 9,$$

$$x^2+5x = 36 \text{ or } -19.$$

Two quadratics of the ordinary form are now presented:

1. $$x^2+5x = 36,$$

$$\therefore x^2+5x+\tfrac{25}{4} = 36+\tfrac{25}{4} = \tfrac{169}{4},$$

$$x = \frac{-5\pm 13}{2} = 4 \text{ or } -9.$$

2. $$x^2+5x = -19.$$

This equation under the test of (255) admits no real root.

The numerical solutions of the equation are therefore

$$x = 4 \text{ and } x = -9.$$

270. $$\sqrt[3]{x}+\sqrt[3]{2x-3} = \sqrt[3]{12(x-1)}.$$

If each side of this equation be raised to the third power

$$3x-3+3\sqrt[3]{(2x-3)x}\,\{\sqrt[3]{x}+\sqrt[3]{2x-3}\} = 12(x-1),$$

$$3\sqrt[3]{(2x-3)x}\,\{\sqrt[3]{x}+\sqrt[3]{2x-3}\} = 9(x-1),$$

and by substitution from the given equation,

$$\sqrt[3]{(2x-3)x} \,.\, \sqrt[3]{12(x-1)} = 3(x-1).$$

Since both sides of the equation are divisible by $\sqrt[3]{x-1}$, the equation is satisfied by,

$$\sqrt[3]{x-1} = 0,$$
$$\therefore\ x-1 = 0,$$
$$x = 1.$$

If x has any other value,

$$\sqrt[3]{(2x-3)x}\sqrt[3]{12} = 3\,(x-1)^{\frac{2}{3}},$$
$$12(2x-3)x = 27(x-1)^2,$$
$$4(2x^2-3x) = 9(x^2-2x+1),$$
$$x^2-6x+9 = 0,$$
$$x-3 = 0,$$
$$x = 3.$$

271. $\sqrt[3]{13x+37}-\sqrt[3]{13x-37} = \sqrt[3]{2}$

If each side of the equation be cubed,

$$13x+37-3(13x+37)^{\frac{2}{3}}(13x-37)^{\frac{1}{3}}$$
$$+3(13x+37)^{\frac{1}{3}}(13x-37)^{\frac{2}{3}}-13x+37 = 2,$$

or $3(13x+37)^{\frac{1}{3}}(13x-37)^{\frac{1}{3}}\left\{(13x+37)^{\frac{1}{3}} - (13x-37)^{\frac{1}{3}}\right\} = 72.$

By substitution from the original equation,

$$3(13x+37)^{\frac{1}{3}}(13x-37)^{\frac{1}{3}}\sqrt[3]{2} = 72 = 3\times 24, \quad \text{(A)}$$

and by cubing each side,

$$2(13x+37)(13x-37) = (24)^3,$$
$$(13x)^2-(37)^2 = 6912,$$
$$(13x)^2 = 8281 = (91)^2.$$
$$\therefore\ 13x = \pm 91,$$
$$x = \pm 7.$$

On substitution for x it will be found that the value 7 satisfies the equation, but the value -7 does not. The latter is therefore to be rejected, but it is instructive to trace the cause of its appearance. It will be found that if the equation had been

$$\sqrt[3]{13x+37}-\sqrt[3]{13x-37} = -\sqrt[3]{2},$$

and if this equation had been treated like that before us, the result marked (A) would have been obtained in this case also. Now the values ± 7 both flow from equation (A). Thus equation (A) replaces two separate equations, and gives the values of x which belong to them, and it is necessary to test by substitution which value belongs to the equation presented.

272. *Examples for Practice.*

Obs.—Real roots only are given as solutions.

1. $3x^2 = 5x^4 - 8x^2 - 306$. *Ans.* $x = \pm 3$.
2. $x^3 - 7x^{\frac{3}{2}} = 8$. *Ans.* $x = 4$ or 1.
3. $x^2 = 21 + \sqrt{x^2-9}$. (243) *Ans.* $x = \pm 5$ or $\pm 3\sqrt{2}$.
4. $\sqrt{x^2+1}+4 = \dfrac{5}{\sqrt{x^2+1}}$. (243) *Ans.* $x = 0$ or $\pm 2\sqrt{6}$.
5. $\dfrac{x-1}{\sqrt{x}-1} = x + \frac{5}{4}$. *Ans.* $x = \frac{1}{4}$.
6. $\sqrt{x^2-2x+9} - \dfrac{x^2}{2} = 3 - x$. *Ans.* $x = 0$ or 2.

273. Simultaneous equations also in two or more unknown quantities may be such as produce quadratics, or equations to be solved in the manner of quadratics. Nothing can be done to describe such methods of solution but to offer some examples with the work in detail.

Ex. 1.
$$\left.\begin{array}{rl} 3(x+y) = 7(x-y) & \quad . \quad . \quad (1) \\ x^2+y^2 = 29 & \quad . \quad . \quad (2) \end{array}\right\}$$

From (1)
$$\begin{aligned} 3x+3y &= 7x-7y, \\ 4x &= 10y, \\ 2x &= 5y, \\ 4x^2 &= 25y^2. \end{aligned}$$

From (2)
$$\begin{aligned} 29\times 4 &= 4x^2+4y^2, \\ &= 25y^2+4y^2, \\ &= 29y^2. \\ \therefore\ y^2 &= 4, \\ y &= \pm 2. \\ \text{Then } x &= \pm 5. \end{aligned}$$

$$\therefore \left.\begin{array}{l} x = 5 \\ y = 2 \end{array}\right\} \text{ or } \left.\begin{array}{l} x = -5 \\ y = -2 \end{array}\right\}$$

are the two pairs of solutions.

274. Ex. 2.
$$x^3-y^3 = a^3, \quad . \quad . \quad (1)$$
$$x^6-y^6 = b^6. \quad . \quad . \quad (2)$$

(2)÷(1) gives
$$x^3+y^3 = \frac{b^6}{a^3}, \quad . \quad . \quad (3)$$

(3)+(1) gives
$$2x^3 = \frac{b^6}{a^3}+a^3 = \frac{b^6+a^6}{a^3},$$
$$\therefore\ x = \frac{\sqrt[3]{b^6+a^6}}{a\sqrt[3]{2}}.$$
$$\text{So } y = \frac{\sqrt[3]{b^6-a^6}}{a\sqrt[3]{2}}.$$

275. Ex. 3.
$$\left.\begin{array}{r} 2x^2+3xy = 26 \\ 3y^2+2xy = 39 \end{array}\right\}.$$

Let $y = kx$,
$$\begin{aligned} \therefore\ x^2(2+3k) &= 26, \\ x^2(3k^2+2k) &= 39. \end{aligned}$$
$$\therefore\ \frac{3k^2+2k}{2+3k} = \tfrac{39}{26} = \tfrac{3}{2}.$$

$$6k^2+4k = 6+9k,$$
$$6k^2-5k = 6,$$
$$k^2-\tfrac{5}{6}k+\tfrac{25}{144} = 1+\tfrac{25}{144} = \tfrac{169}{144},$$
$$k = \frac{5\pm 13}{12} = \tfrac{3}{2} \text{ or } -\tfrac{2}{3}.$$

$\therefore$ (1) if $k = \frac{3}{2}$

$$x^2 = \frac{26}{2+3k} = \frac{26\times 2}{13} = 4,$$
$$x = \pm 2,$$

and then $y = \frac{3}{2}x = \pm 3$.

(2) if $k = -\frac{2}{3}$,

$$x^2 = \frac{26}{2+3k} = \frac{26}{2-2}.$$

$\therefore$ x and y have no numerical values. (29)

276. Ex. 4.

$$x^5-y^5 = 2882, \quad . \quad . \quad (1)$$
$$x-y = 2. \quad . \quad . \quad (2)$$

Compatibly with (2) we may assume

$$\left.\begin{aligned} x &= z+1 \\ y &= z-1 \end{aligned}\right\}.$$

$\therefore$ (1) becomes

$$\begin{aligned} 2882 &= (z+1)^5-(z-1)^5 \\ &= 2\{5z^4+10z^2+1\}, \\ &= 10z^4+2\times 10z^2+2. \end{aligned}$$

$$\therefore\ (10z^2)^2+20\times(10z^2)+100 = 28800+100,$$
$$= 28900 = (170)^2.$$

$$\therefore\ 10z^2 = -10\pm 170 = 160, \text{ or } -180,$$
$$z^2 = 16, \text{ or } -18.$$

Rejecting the latter value if we are to obtain numerica results (33) we have

$$z^2 = 16,$$
$$z = \pm 4.$$
$$\therefore\ \left.\begin{aligned} x &= 5, \text{ or } -3 \\ y &= 3, \text{ or } -5 \end{aligned}\right\}.$$

277. Ex. 5.
$$\frac{1}{x^2}+\frac{1}{y^2}=5, \quad . \quad . \quad (1)$$
$$\frac{1}{x}-\frac{1}{y}=1. \quad . \quad . \quad (2)$$

Consistently with (2) we may assume
$$\frac{1}{x}=\frac{z+1}{2}, \quad \frac{1}{y}=\frac{z-1}{2},$$
the difference of these quantities being 1.

Then equation (1) gives
$$\frac{(z+1)^2}{4}+\frac{(z-1)^2}{4}=5,$$
$$\text{or } \frac{z^2+1}{2}=5,$$
$$z^2+1=10,$$
$$z^2=9,$$
$$z=\pm 3.$$
$$\therefore \frac{1}{x}=\frac{1\pm 3}{2}=2, \text{ or } -1,$$
$$x=\tfrac{1}{2}, \text{ or } -1,$$
$$\frac{1}{y}=\frac{\pm 3-1}{2}=1, \text{ or } -2,$$
$$y=1, \text{ or } -\tfrac{1}{2}.$$

278. Ex. 6.
$$4x^2+4y^2=17xy, \quad . \quad . \quad (1)$$
$$\sqrt{x}-\sqrt{y}=3. \quad . \quad . \quad (2)$$

Equation (1) will give the value of $\frac{x}{y}$, and then (2) will give the separate values of x and y.

Equation (1) is
$$\left(\frac{x}{y}\right)^2-\tfrac{17}{4}\cdot\frac{x}{y}+(\tfrac{17}{8})^2=-1+(\tfrac{17}{8})^2=\frac{255}{64},$$
$$\text{whence } \frac{x}{y}=\frac{17\pm 15}{8}=4, \text{ or } \tfrac{1}{4},$$

$$\therefore \sqrt{\frac{x}{y}} = \pm 2, \text{ or } \pm \tfrac{1}{2}.$$

Hence (2) gives

$$1. \quad \frac{3}{\sqrt{y}} = \sqrt{\frac{x}{y}} - 1 = \pm 2 - 1 = 1, \text{ or } -3.$$

$$\therefore \quad y = 9, \text{ or } 1, \qquad \therefore \; x = 36, \text{ or } 4.$$

$$2. \quad \frac{3}{\sqrt{y}} = \pm\tfrac{1}{2} - 1 = -\tfrac{1}{2}, \text{ or } -\tfrac{3}{2}.$$

$$\therefore \quad y = 36, \text{ or } 4, \qquad \therefore \; x = 9, \text{ or } 1.$$

Hence the pairs of solutions are

$$\left.\begin{array}{l} x = 36 \\ y = 9 \end{array}\right\} \left.\begin{array}{l} x = 4 \\ y = 1 \end{array}\right\} \left.\begin{array}{l} x = 9 \\ y = 36 \end{array}\right\} \left.\begin{array}{l} x = 1 \\ y = 4 \end{array}\right\} (243).$$

279. Ex. 7. $\left.\begin{array}{r} (ax+by)^2 + (ay-bx)^2 = 2\left(\dfrac{a}{b}+\dfrac{b}{a}\right)^2, \\ \dfrac{x}{y}+\dfrac{y}{x} = 2\dfrac{a^2+b^2}{a^2-b^2}. \end{array}\right\}$

The former equation gives

$$a^2(x^2+y^2) + b^2(x^2+y^2) = 2\frac{(a^2+b^2)^2}{a^2b^2},$$

$$\text{or } x^2+y^2 = 2\frac{a^2+b^2}{a^2b^2} \quad . \quad (1)$$

The latter equation gives

$$x^2+y^2 = 2\frac{a^2+b^2}{a^2-b^2}xy, \quad . \quad . \quad (2)$$

$$\therefore \frac{a^2+b^2}{a^2b^2} = \frac{a^2+b^2}{a^2-b^2}xy,$$

$$xy = \frac{a^2-b^2}{a^2b^2},$$

$$2xy = 2\frac{a^2-b^2}{a^2b^2}. \quad . \quad . \quad (3)$$

(1) + (3) gives

$$x^2 + 2xy + y^2 = \frac{4a^2}{a^2b^2},$$

$$\text{or } x+y = \pm\frac{2}{b}.$$

$$(1)-(3) \text{ gives} \quad x^2-2xy+y^2 = \frac{4b^2}{a^2b^2},$$

$$\text{or } x-y = \pm\frac{2}{a}.$$

$$\therefore\ x = \pm\frac{1}{a}\pm\frac{1}{b}.$$

$$y = \pm\frac{1}{a}\mp\frac{1}{b},$$

or the four pairs of solutions result

$$\left.\begin{aligned} x &= \frac{1}{a}+\frac{1}{b} \\ y &= \frac{1}{a}-\frac{1}{b} \end{aligned}\right\} \quad \left.\begin{aligned} x &= \frac{1}{a}-\frac{1}{b} \\ y &= \frac{1}{a}+\frac{1}{b} \end{aligned}\right\} \quad \left.\begin{aligned} x &= -\frac{1}{a}-\frac{1}{b} \\ y &= -\frac{1}{a}+\frac{1}{b} \end{aligned}\right\} \quad \left.\begin{aligned} x &= -\frac{1}{a}+\frac{1}{b} \\ y &= -\frac{1}{a}-\frac{1}{b} \end{aligned}\right\}.$$

280. Ex. 8. $\quad \dfrac{(x+y)^2}{a^2}+\dfrac{(x-y)^2}{b^2} = 8 \quad . \quad . \quad (1)$

$$x^2+y^2 = 2(a^2+b^2) \quad . \quad (2)$$

$$\text{Since} \quad 2(x^2+y^2) = (x+y)^2+(x-y)^2$$

the second equation may be replaced by the form

$$(x+y)^2+(x-y)^2 = 4(a^2+b^2) \quad . \quad (3)$$

It will now be seen that this new form of (2) and the equation (1) present us with two equations of simple form if $(x+y)^2$, and $(x-y)^2$ are regarded as the unknown quantities.

$$(3)\div b^2 \text{ gives } \frac{(x+y)^2}{b^2}+\frac{(x-y)^2}{b^2} = 4\left(1+\frac{a^2}{b^2}\right),$$

and when this is subtracted from (1)

$$(x+y)^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right) = 4-4\frac{a^2}{b^2},$$

$$(x+y)^2\frac{b^2-a^2}{a^2b^2} = 4\frac{b^2-a^2}{b^2}.$$

$$\therefore\ (x+y)^2 = 4a^2.$$

$$\text{So } (x-y)^2 = 4b^2.$$

$$\therefore\ \left.\begin{aligned} x+y &= \pm 2a \\ x-y &= \pm 2b \end{aligned}\right\}.$$

$$\therefore\quad 2x = \pm 2a \pm 2b.$$

$$\left.\begin{aligned} x &= \pm a \pm b \\ y &= \pm a \mp b \end{aligned}\right\}.$$

So that the following four pairs of solutions eventually arise :

$$\left.\begin{aligned} x &= a+b \\ y &= a-b \end{aligned}\right\},$$

$$\left.\begin{aligned} x &= a-b \\ y &= a+b \end{aligned}\right\},$$

$$\left.\begin{aligned} x &= -a-b \\ y &= -a+b \end{aligned}\right\},$$

$$\left.\begin{aligned} x &= -a+b \\ y &= -a-b \end{aligned}\right\}.$$

281. Ex. 9. $\sqrt{y+x}+\sqrt{y-x} = \sqrt{y^2-x^2}$, (1)

$$2(y-x) = x^2. \qquad (2)$$

The first equation may be expected to give a relation between x and y, and then the second will determine their separate values.

If both sides of (1) be squared,

$$2y+2\sqrt{y^2-x^2} = y^2-x^2,$$

$$y^2-x^2-2\sqrt{y^2-x^2}+1 = 2y+1,$$

$$\sqrt{y^2-x^2} = 1\pm\sqrt{2y+1}.$$

But from (2),

$$2y+1 = x^2+2x+1 = (x+1)^2,$$

$$\therefore\ \sqrt{y^2-x^2} = 1\pm(x+1) = x+2 \text{ or } -x,$$

$$y^2-x^2 = x^2+4x+4 \text{ or } x^2.$$

1. $$\frac{y^2}{2} = x^2+2x+1 = (x+1)^2.$$

$$\therefore \frac{y^2}{2} = 2y + 1,$$

$$y^2 - 4y + 4 = 6,$$

whence $y = 2 \pm \sqrt{6}$, $x = -1 \pm (\sqrt{3} \pm \sqrt{2})$.

2. $$y^2 = 2x^2.$$

Since $x^2 + 2x = 2y$,

$$(x^2 + 2x)^2 = 4y^2 = 8x^2,$$

$$\therefore \quad x = 0, \quad \text{then } y = 0;$$

or $$(x+2)^2 = 8,$$

$$x + 2 = \pm 2\sqrt{2},$$

$$x = -2 \pm 2\sqrt{2}.$$

and $$y^2 = 2\,(12 \mp 8\sqrt{2}),$$

$$= 4\,(6 \mp 4\sqrt{2}),$$

$$y = \pm 2\sqrt{6 \mp 4\sqrt{2}}.$$

282. Ex. 10. $$\frac{\sqrt[3]{x+y} - \sqrt[3]{x-y}}{\sqrt[3]{x+y} + \sqrt[3]{x-y}} = 3, \quad (1)$$

$$x^2 + y^2 = 75. \quad (2)$$

From (1) we have,

$$\sqrt[3]{x+y} - \sqrt[3]{x-y} = 3\sqrt[3]{x+y} + 3\sqrt[3]{x-y},$$

$$\sqrt[3]{x+y} = -2\sqrt[3]{x-y}.$$

If both sides of this equation be cubed,

$$x + y = -8(x-y),$$

or $$9x = 7y,$$

$$y = \frac{9x}{7}.$$

Then in (2)

$$x^2 + (\tfrac{9}{7})^2 x^2 = 75,$$

$$x^2 = \tfrac{15}{26} \times 49.$$

$$\therefore x = \pm 7\sqrt{\tfrac{15}{26}}.$$

$$\text{Hence } y = \pm 9\sqrt{\tfrac{15}{26}},$$

the double signs being used dependently, the upper and the lower being taken together, so that two pairs of values result.

283. Ex. 11.
$$\left.\begin{aligned} \frac{x^3}{y}-\frac{y^3}{x} &= \tfrac{65}{6}, \\ \frac{x}{y}-\frac{y}{x} &= \tfrac{5}{6}. \end{aligned}\right\}$$

The equations can be altered into the forms

$$\left.\begin{aligned} x^4-y^4 &= \tfrac{65}{6}xy, \quad (1) \\ x^2-y^2 &= \tfrac{5}{6}xy, \quad (2) \end{aligned}\right\}$$

$$\begin{aligned} \therefore\ (1)\div(2) \text{ gives } x^2+y^2 &= 13, \\ 4x^2y^2 &= (x^2+y^2)^2-(x^2-y^2)^2, \\ &= 169-\tfrac{25}{36}x^2y^2. \end{aligned}$$

$$\begin{aligned} \therefore\ 4x^2y^2+\tfrac{25}{36}x^2y^2 &= 169, \\ \frac{169x^2y^2}{36} &= 169, \\ x^2y^2 &= 36, \\ xy &= \pm 6. \end{aligned}$$

1. Let $xy = 6$.

$$\begin{aligned} \therefore\ x^2-y^2 &= 5, \text{ from } (2) \\ \text{while } x^2+y^2 &= 13. \end{aligned}$$

$$\begin{aligned} \therefore\ 2x^2 &= 18,\ x^2 = 9,\ x = \pm 3, \\ 2y^2 &= 8,\ \ y^2 = 4,\ y = \pm 2. \end{aligned}$$

Although the double signs in the values of x and y appear to arise independently, they are governed by the condition that xy is positive, or that x and y have the same sign. Hence we may have

$$\begin{aligned} x &= 3 \\ y &= 2 \end{aligned} \text{ or } \left.\begin{aligned} x &= -3 \\ y &= -2 \end{aligned}\right\}.$$

2. Let $xy = -6$.

$$\begin{aligned} \therefore\ x^2-y^2 &= -5, \\ \text{while } x^2+y^2 &= 13. \\ \therefore\ 2x^2 &= 8,\ x = \pm 2, \\ 2y^2 &= 18,\ y = \pm 3. \end{aligned}$$

Here the values to be selected must be such that x and y have contrary signs, since $xy = -6$,

$$\therefore \left.\begin{array}{l} x = 2 \\ y = -3 \end{array}\right\} \text{ or } \left.\begin{array}{l} x = -2 \\ y = 3 \end{array}\right\}.$$

The solution of this equation supplies an important caution, because it shows how values may sometimes arise by the process of solution adopted, which are not admissible as roots of the proposed equation. Thus in the subdivision (1), the equations at which we have arrived,

$$\left.\begin{array}{l} x^2-y^2 = 5 \\ x^2+y^2 = 13 \end{array}\right\}, \quad \text{(A)}$$

include the two pairs of solutions of the proposed equation, but include two other pairs besides, namely,

$$\left.\begin{array}{l} x = 3 \\ x = -2 \end{array}\right\} \quad \left.\begin{array}{l} x = -3 \\ x = 2 \end{array}\right\},$$

which are not admissible; the reason being that equations (A), as far as they represent the proposed equation, do so only under the limitation of xy being positive; and by this limitation these latter pairs of values of x and y are rejected.

Thus four pairs of values of x and y result as solutions of the proposed equation.

284. Ex. 12.
$$\left.\begin{array}{lr} x^4+x^2y^2+y^4 = 108x^2 \quad . \quad . \quad . & (1) \\ x^2+xy +y^2 = 18x \quad . \quad . \quad . & (2) \end{array}\right\}$$

The key to these equations is the fact that

$$x^4+x^2y^2+y^4 = (x^2+xy+y^2)(x^2-xy+y^2).$$

$(1) \div (2)^2$ gives

$$\frac{108}{(18)^2} \text{ or } \tfrac{1}{3} = \frac{x^4+x^2y^2+y^4}{(x^2+xy+y^2)^2} = \frac{x^2-xy+y^2}{x^2+xy+y^2}.$$

$$x^2+xy+y^2 = 3(x^2-xy+y^2),$$

$$2(x^2+y^2) = 4xy,$$

$$x^2-2xy+y^2 = 0,$$

$$x-y = 0,$$

$$x = y.$$

Then from (2),

$$3x^2 = 18x,$$

whence either $x = 0$,

or $3x = 18,\ x = 6.$

$$\therefore \left.\begin{array}{l} x = 0 \\ y = 0 \end{array}\right\} \quad \left.\begin{array}{l} x = 6 \\ y = 6 \end{array}\right\}$$

are the solutions.

The following is another example to which the same treatment will apply:

$$\left.\begin{array}{r} x^2+xy+y^2 = 49 \\ x^4+x^2y^2+y^4 = 931 \end{array}\right\}.$$

285. Ex. 13.

$$\left.\begin{array}{rll} xz = y^2 & . \quad . & (1) \\ x+y+z = 13 & . \quad . & (2) \\ x^2+y^2+z^2 = 91 & . \quad . & (3) \end{array}\right\}.$$

Since in (1) $\frac{x}{y} = \frac{y}{z}$, we may represent each of these fractions by the symbol k, so that

$$\left.\begin{array}{r} x = ky \\ y = kz \end{array}\right\}.$$

$$\therefore\ x = k^2z.$$

$\therefore$ (2) becomes $(k^2+k+1)z = 13,$

whence $(k^2+k+1)^2z^2 = 169;$

and (3) becomes $(k^4+k^2+1)z^2 = 91.$

$$\therefore\ \tfrac{169}{91} \text{ or } \tfrac{13}{7} = \frac{(k^2+k+1)^2}{k^4+k^2+1},$$

$$= \frac{k^2+k+1}{k^2-k+1}.$$

$$\therefore\ 13(k^2-k+1) = 7(k^2+k+1)$$

$$6k^2-20k = -6.$$

$$k^2 - \tfrac{10}{3}k + \tfrac{25}{9} = -1 + \tfrac{25}{9} = \tfrac{16}{9}.$$

$$\therefore\ k = \frac{5 \pm 4}{3} = 3, \text{ or } \tfrac{1}{3}.$$

$$\therefore\ 1+k+k^2 = 1+3+9 = 13,$$

$$\text{or } 1+\tfrac{1}{3}+\tfrac{1}{9} = \tfrac{13}{9}.$$

$$\therefore\ z = \frac{13}{1+k+k^2} = 1, \text{ or } 9.$$

$$y = kz = 3,$$

$$x = ky = 9, \text{ or } 1.$$

$\therefore$ the two sets of values which satisfy the equations are

$$\left.\begin{array}{l} x = 9 \\ y = 3 \\ z = 1 \end{array}\right\} \qquad \left.\begin{array}{l} x = 1 \\ y = 3 \\ z = 9 \end{array}\right\}.$$

286. Ex. 14.

$$\left.\begin{array}{rll} x^2+y^2+z^2 = 50 & . \quad . & (1) \\ x-y-z = -6 & . \quad . & (2) \\ x(y+z) = 27 & . \quad . & (3) \end{array}\right\}.$$

Equations (2) and (3) are two equations in the two quantities x and $y+z$, and will therefore suffice for determining these quantities.

(2) gives $\qquad y+z = x+6.$

$\therefore$ (3) becomes $x(x+6) = 27,$

$$x^2+6x+9 = 36,$$

$$x = -3\pm 6 = 3, \text{ or } -9.$$

$$\therefore\ y+z = 9, \text{ or } -3.$$

$$\therefore\ y^2+z^2 = 50-x^2 = 41, \text{ or } -41.$$

The latter value is inadmissible if numerical results are required.

$$\therefore\ 82 = 2(y^2+z^2),$$

$$= (y-z)^2+(y+z)^2.$$

$$\therefore\ (y-z)^2 = 82-81, \text{ or } 82-9,$$

$$= 1, \text{ or } 73.$$

$$y-z = \pm 1, \text{ or } \pm\sqrt{73},$$

and $\quad y+z = 9.$

$$\therefore\ 2y = 9\pm 1, \text{ or } 9\pm\sqrt{73},$$

$$y = 5, \text{ or } 4, \text{ or } \tfrac{1}{2}(9\pm\sqrt{73}),$$

$$2z = 9\mp 1, \text{ or } 9\mp\sqrt{73}.$$

$$\therefore\ z = 4, \text{ or } 5, \text{ or } \tfrac{1}{2}(9\mp\sqrt{73}).$$

287. *Examples for Practice.*

1. $\left.\begin{aligned} x^2+xy &= a^2 \\ y^2+xy &= b^2 \end{aligned}\right\}.$ *Ans.* $x = \pm\dfrac{a^2}{\sqrt{a^2+b^2}}$, $y = \pm\dfrac{b^2}{\sqrt{a^2+b^2}}$.

2. $x+\dfrac{1}{y} = y+\dfrac{1}{x} = \frac{5}{2}$. *Ans.* $x = y = 2$, $x = y = \frac{1}{2}$.

3. $\left.\begin{aligned} x^2+y &= 4 \\ y^2+x &= 10 \end{aligned}\right\}.$ *Ans.* $x = 1$, $y = 3$.

4. $\left.\begin{aligned} x+\frac{1}{y} &= 5\cdot5 \\ y+\frac{1}{x} &= 2\cdot5 \end{aligned}\right\}.$ *Ans.* $x = \dfrac{55 \pm \sqrt{2145}}{20}$, $y = \dfrac{5\sqrt{11} \pm \sqrt{195}}{4\sqrt{11}}$.

5. $\left.\begin{aligned} x^2-y^2 &= 7 \\ xy &= 12 \end{aligned}\right\}.$ *Ans.* $x = \pm4$, $y = \pm3$.

6. $\left.\begin{aligned} x^2+y^2 &= 25 \\ xy &= 12 \end{aligned}\right\}.$ *Ans.* $x = \pm4$, $y = \pm3$.

7. $\left.\begin{aligned} 3x+2y &= 13 \\ xy &= 6 \end{aligned}\right\}.$ *Ans.* $x = 3$, $y = 2$.

8. $\left.\begin{aligned} 4x^2+7y^2 &= 148 \\ 3x^2-\ y^2 &= 11 \end{aligned}\right\}.$ *Ans.* $x = \pm3$, $y = \pm4$.

9. $\left.\begin{aligned} 3x^2+2y^2 &= 30 \\ 5x^2-\ y^2 &= 11 \end{aligned}\right\}.$ *Ans.* $x = \pm2$, $y = \pm3$.

10. $\left.\begin{aligned} \frac{x}{2}+\frac{y}{3} &= 1 \\ \frac{2}{x}+\frac{3}{y} &= 4 \end{aligned}\right\}.$ *Ans.* $x = 1$, $y = \frac{3}{2}$.

11. $\left.\begin{aligned} 2x+3y &= 4 \\ \frac{1}{2x}+\frac{1}{3y} &= 1 \end{aligned}\right\}.$ *Ans.* $x = 1$, $y = \frac{2}{3}$.

12. $\left.\begin{aligned} x+y &= c \\ \frac{a^2}{x}+\frac{b^2}{y} &= \frac{(a+b)^2}{c} \end{aligned}\right\}.$ *Ans.* $x = \dfrac{ac}{a+b}$, $y = \dfrac{bc}{b+c}$.

13. $\left.\begin{aligned} x^4+y^4 &= 337 \\ xy &= 12 \end{aligned}\right\}.$ *Ans.* $x = \pm 4,\ y = \pm 3,$
$x = \pm 3,\ y = \pm 4.$

14. $\left.\begin{aligned} x^2+2ax+y^2 &= (a+b)^2 \\ x-b &= y-a \end{aligned}\right\}.$

Ans. $x = -a+\frac{b}{2}\pm\sqrt{a^2+ab+\frac{b^2}{4}},$
$y = -\frac{b}{2}\pm\sqrt{a^2+ab+\frac{b^2}{4}}.$

15. $x+y+3\sqrt{x+y} = x^2+y^2 = 10.$ *Ans.* $x = 3,\ y = 1.$

16. $\left.\begin{aligned} \frac{x^2}{y^2}+\frac{y^2}{x^2}+\frac{x}{y}-\frac{y}{x} &= \frac{106}{9} \\ xy &= 3 \end{aligned}\right\}.$

Ans. $x = \pm 3,$ $y = \pm 1,$
$x = \pm\sqrt{\frac{\sqrt{157}-11}{2}},\ y = \pm\sqrt{\frac{\sqrt{157}+11}{2}}.$

17. $\left.\begin{aligned} \frac{x+y}{7} &= \frac{8}{x+y+1} \\ xy &= 12 \end{aligned}\right\}.$ *Ans.* $x = 4,\ y = 3,$
$x = 3,\ y = 4,$
$x = -2,\ y = -6,$
$x = -6,\ y = -2.$

18. $\left.\begin{aligned} \sqrt{x^2-11}+\sqrt{y^2-5} &= 7 \\ x^2y^2-11y^2-5x^2 &= 45 \end{aligned}\right\}.$ *Ans.* $x = \pm 6,$
$y = \pm 3,$
$x = \pm\sqrt{15},$
$y = \pm\sqrt{30}.$

19. Prove that the following equations are incompatible:

$$\left.\begin{aligned} \frac{x+\sqrt{x^2-y^2}}{x-\sqrt{x^2-y^2}}+\frac{x-\sqrt{x^2-y^2}}{x+\sqrt{x^2-y^2}} &= 9 \\ \frac{\sqrt{x^2-y^2}+y}{\sqrt{x^2-y^2}-y}-\frac{\sqrt{x^2-y^2}-y}{\sqrt{x^2-y^2}+y} &= 7 \end{aligned}\right\}.$$

20. $x(y+z) = 5,\ y(x+z) = 8,\ z(x+y) = 9.$

Ans. $x = 1,\quad y = 2,\quad z = 3;$
or $x = -1,\ y = -2,\ z = -3.$

21. $\left.\begin{aligned} x^2-y^2+z^2 &= 6 \\ 2yz-zx+2xy &= 13 \\ x-y+z &= 2 \end{aligned}\right\}$ *Ans.* $x = 1,\ y = 2,\ z = 3;$
or $x = 3,\ y = 2,\ z = 1.$

288. *Problems producing Quadratic Equations.*

Problems may produce quadratic equations, either single or simultaneous, as it may be needful to assume one unknown quantity or more than one. The conditions of the problem being thus expressed in Algebra, the solution of the equation or equations supplies the required answer in the manner which has been already exemplified.

Since a quadratic equation has generally two distinct roots, two answers may thus appear to arise, but both will not always belong to the problem on which the equation is formed.

Suppose the problem be this: What is the positive integer such that if it be squared and 12 be added to its square, the result is seven times the number?

If x be taken to represent the number, the property which is to distinguish and define it is expressed in Algebra by the equation

$$x^2+12 = 7x.$$

From this we find $x = 3$ or $x = 4$. Two answers to the question are thus supplied, and it will be found on trial that either of them fulfils the condition expressed in the statement of the problem.

289. Again: What is the number such that if it be squared and 9 be added to the square, the result is six times the number? The resulting equation is

$$x^2+9 = 6x,$$

whence only one value arises, $x = 3$, the equation having equal roots.

290. If another instance be, What is the positive integral number such that if 12 be subtracted from the double of its square the result is five times the number? the equation expressing this property is

$$2x^2-12 = 5x,$$

$$\text{whence } x = 4 \text{ or } -\tfrac{3}{2}.$$

Hence 4 is the number answering the conditions of the

problem, the other root $-\frac{3}{2}$ being inapplicable, because a positive integer is required.

It is obvious in this instance that $-\frac{3}{2}$ is an answer which cannot be admitted, and that the other root is the only one to fulfil the conditions of the problem, but we have to account for the appearance of such an inadmissible result, if the reasoning from the premises has been correctly performed. The explanation of its appearance is this, that the algebraical statement of the question in the equation is more comprehensive than the verbal statement in the enunciation of the question. The problem asks for a positive integer. When x is taken as the symbol to represent it, x cannot be tied to mean nothing but a positive integer, and accordingly must mean every number positive or negative, integral or fractional, which has the property that $2x^2-12$ is equal to $5x$. There are two such numbers, it appears, 4 and $-\frac{3}{2}$, and the equation by its roots presents them both.

291. To take another instance where the algebraic statement of a question expresses more than the fact of the question which immediately produces it, let this be the problem considered.

Two rectangles have the same size, 18 square feet. The length of one is 3 feet more than the length of the other, and their breadths together make 5 feet. What are their dimensions?

Let x be the length of one of the rectangles in feet, and $x+3$ the length of the other.

Then $\frac{18}{x}$ and $\frac{18}{x+3}$ are their respective breadths in feet.

Hence the condition of the question is that

$$\frac{18}{x}+\frac{18}{x+3}=5, \quad \text{(A)}$$

when $x=6$ or $-\frac{9}{5}$.

The former root gives the solution of the problem, making the lengths of the rectangles 6 and 9 feet, their breadths accordingly 3 and 2 feet. The latter root $-\frac{9}{5}$ gives

no result applicable to a rectangle, but it has arisen because the equation (A) expresses not only the condition of the particular problem before us, but expresses and includes all values of x such that $\frac{18}{x}$ and $\frac{18}{x+3}$ together make 5. Now $-\frac{9}{5}$ is a value of x which makes these two fractions added together give 5, and accordingly rises before us out of the solution of the equation.

292. Let the problem be this : What is the age of a child when, if the number of his years be squared and 81 added, the result will be thirty times the number of his years.

Let the child be x years old, then

$$x^2 + 81 = 30x,$$

$$\text{when } x = 3 \text{ or } 27.$$

The latter root is inadmissible, because the years of a *child's* age are required, and 3 years accordingly is his age.

Now the value $x = 27$ arises, because in taking x to represent the age, it is impossible to restrict the symbol to a *child's* age. We must accept whatever numbers result fulfilling the terms of the equation, which is thus an expression in Algebra of a fact respecting numbers more comprehensive than the statement of the problem in words.

293. The following problem is one which admits no solution, and Algebra will declare for us that there is none.

What is the number such that if it be squared and 5 added to the square the result is four times the number?

When x is supposed to represent the number,

$$x^2+5 = 4x.$$

$$\therefore x^2-4x+4 = -1,$$

$$(x-2)^2 = -1.$$

Now no numerical value of x fulfils this condition, the equation having unreal roots, as the test of (255) would have shown before the solution was commenced. Hence no number fulfils the conditions of the problem.

In this case it is not that there may be a number to fulfil the terms of the question, and that Algebra admits its inability

to discover what that number is, or whether there be one, but Algebra demonstrates that there is no such number. The reasoning is this: A number under the proposed condition does or does not exist. If it exists let x represent it. Correct reasoning thus leads to an impossibility. Therefore the supposition of the number existing is wrong.

294. The length of a rectangular field exceeds the breadth by a yard, and the area is 10,100 square yards. Find the length of either side.

Let the length be $x + 1$ yards, and the breadth x yards.

Hence the area is $x(x+1)$ square yards.

$$\therefore\ x(x+1) = 10,100.$$

This equation may be solved by the ordinary method as a quadratic, and the results are $x = 100$ and -101. After the latter is rejected as incompatible with the meaning of the problem, the dimensions of the field are pronounced to be 101 yards in length and 100 yards in breadth.

The following problems are similar:

Find the length and breadth of a room of 195 square feet, where the length is 2 feet more than the breadth.

Ans. The length is 15 feet, the breadth 13 feet.

Would the area be the same if the length were reduced 1 foot and the breadth enlarged 1 foot?

A brigade is marching in column with 5 men more in depth than in front, but if they form 5 deep they increase the front by 845 men. How many men were there in the brigade? *Ans.* 4550 men.

295. In a concert room 800 persons are seated on benches of equal length. If there were 20 fewer benches, it would be necessary that two persons more should sit on each bench. Find the number of benches.

Suppose that $x-1$ persons sit on each bench in the first case.

Then there are $\frac{800}{x-1}$ benches.

In the second case $x+1$ persons sit on each bench, and there are accordingly $\frac{800}{x+1}$ benches.

In the latter case, the question states, there are 20 fewer benches than in the former.

$$\therefore \frac{800}{x-1} - \frac{800}{x+1} = 20,$$

$$40\left\{\frac{1}{x-1} - \frac{1}{x+1}\right\} = 1,$$

$$\frac{80}{x^2-1} = 1$$

$$x^2 - 1 = 80,$$

$$x^2 = 81,$$

$$x = \pm 9.$$

The negative sign giving a negative number of persons is inadmissible, and therefore there are in the first case $x-1$ or 8 persons on each bench and 100 benches, in the second case 10 on each bench and 80 benches.

296. An article is sold for 9*l.* at a loss of as much per cent. as it is worth. Find its value.

Let x*l.* be the value.

The loss on the sale is $x-9$*l.*

Now this is x per cent. of its value, or $\frac{x}{100}$ parts of x*l.*

$$\therefore x - 9 = \frac{x^2}{100},$$

$$x = 90 \text{ or } 10,$$

$\therefore$ 90*l.* or 10*l.* is the value.

The following problem is similar:

An article is sold at a loss of as much per cent. as it is worth in pounds. Show that it cannot be sold for more than 25*l.* (255)

297. A wine merchant sold 7 dozen of sherry and 12 dozen of claret for 50*l.*, and found that he had sold 3 dozen more of sherry for 10*l.* than of claret for 6*l.* What was the price of each?

Let a dozen of sherry be sold for x*l.*

„ claret „ y*l.*

Then 10*l.* will purchase $\frac{10}{x}$ dozen of sherry.

6*l.* „ $\frac{6}{y}$ „ claret.

Therefore the second condition gives

$$\frac{10}{x} = \frac{6}{y} + 3 \quad . \quad . \quad . \quad (1)$$

while the first condition gives

$$7x + 12y = 50$$

$$\text{or } 7x = 50 - 12y \quad . \quad . \quad (2)$$

$\therefore$ (1) $\times$ (2) gives

$$70 = (50 - 12y)\left(\frac{6}{y} + 3\right),$$

$$\text{whence } 9y^2 - 2y = 75,$$

$$y^2 - \tfrac{2}{9}y + \tfrac{1}{81} = \tfrac{1}{81} + \tfrac{75}{9} = \tfrac{676}{81},$$

$$y = \frac{1 \pm 26}{9} = 3 \text{ or } -\tfrac{25}{9}.$$

$$\text{Then } x = 2 \text{ or } \tfrac{250}{21}.$$

The nature of the question excludes the negative root, and the solution of the question is that a dozen of sherry is sold for 2*l.* and a dozen of claret for 3*l.*

298. A cask originally full of wine is filled up with water after three gallons of wine have been drawn from it. Four gallons of the mixture are now drawn off, and when the cask has been again filled up with water it contains equal

quantities of wine and water. Find the contents of the cask.

Let $2x$ gallons be the contents of the cask. Then at the end of the operations described there will be x gallons of wine in it.

After the first filling up there are in the cask

$$2x-3 \text{ gallons of wine,}$$
$$3 \quad ,, \quad \text{water.}$$

Hence in every gallon of liquid that is drawn out there are $\frac{2x-3}{2x}$ gallons of wine.

$\therefore$ in drawing off 4 gallons of liquid $4 . \frac{2x-3}{2x}$ gallons of wine are withdrawn.

$\therefore$ the wine finally left is $2x-3-4\,\frac{2x-3}{2x}$ gallons, and this by the condition of the question is x gallons.

$$\therefore\ 2x-3-4\,\frac{2x-3}{2x} = x, \qquad \text{(A)}$$
$$2x^2-3x-4x+6 = x^2,$$
$$x^2-7x+\tfrac{49}{4} = \tfrac{49}{4}-6 = \tfrac{25}{4},$$
$$x = \frac{7 \pm 5}{2} = 6 \text{ or } 1.$$

Hence $\qquad 2x = 12, \text{ or } 2.$

The answer therefore to this question is that the cask contains 12 gallons. The value $2x = 2$ is inconsistent with the drawing off of 3 gallons, but the equation (A) can embody no such restriction upon the value of x.

299. The sum of two numbers is 22, and the sum of their cubes 2926. Find them.

If we were to suppose x to be one of these numbers, and $22-x$ the other, it will be found that more laborious numbers arise than when the following is the assumption made.

Let $11+x$ and $11-x$ be the numbers, their sum accordingly being 22.

$$\therefore\ 2926 = (11+x)^3+(11-x)^3,$$
$$3x^2 = 12,$$
$$x^2 = 4,$$
$$x = \pm 2.$$

$\therefore$ the numbers are 13 and 9, whichever sign x is allowed to take.

300. What are the two numbers whose sum multiplied by the greater is 204, and whose difference multiplied by the less is 35.

Let x be the greater and y the less of the two numbers. By the conditions of the question

$$(x+y)x = 204 \quad . \quad . \quad (1)$$
$$(x-y)y = 35 \quad . \quad . \quad (2)$$

Let $y = kx$.

$$\therefore\ (k+1)x^2 = 204,$$
$$(1-k)kx^2 = 35.$$
$$\therefore\ \frac{k+1}{k-k^2} = \tfrac{204}{35}.$$

Hence $k = \frac{7}{17}$, or $\frac{5}{12}$.

(1) If $k = \frac{7}{17}$, $k+1 = \frac{24}{17}$

$$x^2 = \frac{17 \times 204}{24},$$
$$x = \pm\frac{17}{\sqrt{2}},$$
$$y = \pm\frac{7}{\sqrt{2}}.$$

Since integral numbers are supposed to be required, these values are inadmissible.

(2) If $k = \frac{5}{12}$

$$x = \pm 12, \quad y = \pm 5,$$

$\therefore$ 12 and 5 are the numbers required.

The following problem is similar:

The product of two numbers is 24, and their sum multiplied by their difference is 20. What are the numbers?

Ans. The numbers are 6 and 4.

301. *Problems for Practice.*

1. What are the two numbers whose difference is 5, and their sum multiplied by the greater is 228? *Ans.* 7 and 12.

2. Find three consecutive numbers whose product is three times the middle number. *Ans.* The numbers are 1, 2, and 3.

3. A certain number consists of two digits. The left hand digit is double of the right hand digit, and if the digits be inverted the product of the number thus formed and the original number is 2268. Find the number.

Ans. The number is 63.

4. A farmer bought some sheep for 72*l.*, and found that if he had received 6 more for the same money he would have paid 1*l.* less for each. What was the number of sheep bought? *Ans.* 18 sheep.

5. Two trains start at the same time to run 1200 miles. One runs 10 miles an hour faster than the other, and arrives 10 hours sooner. What was the speed of each, supposing it to be uniform?

Ans. One 30 miles and the other 40 miles an hour.

6. Goods are marked for sale at a price which gives a profit of n per cent. on the selling price and $n + 50$ per cent. on the cost price. Find n. *Ans.* $n = 50$.

7. The longest side or hypotenuse of a right-angled triangle is 10 yards, and of the other sides one is two yards longer than the other. Find the lengths of these sides.

Ans. They are 6 and 8 yards long (*Euclid*, I. 47).

8. Two lengths of cloth are bought for 4*l.* 9*s.* One is three yards longer than the other, and each cost as many shillings the yard as it is yards in length. What are their lengths? *Ans.* The lengths are 5 and 8 yards.

9. Two men run at uniform paces along a line, one from A to B, the other from B to A. Starting together from the points A and B respectively they meet after 6 minutes, and one performs the whole distance in $3\frac{1}{2}$ minutes less than the other. Show that the pace of the slower one is $\frac{3}{4}$ of the *pace* of the other.

CHAPTER VIII.

RATIO AND PROPORTION.

302. *Def.*—The fraction $\frac{a}{b}$ is the expression and measure of the ratio which the quantity a has to the quantity b.

This relation is expressed in symbol by the form $a : b$, but the measure of its value is the fraction $\frac{a}{b}$.

It will be observed that this definition agrees with, and includes, the usual arithmetical conception of the ratio of two quantities which admit of definite arithmetical expression to some common unit, or are, as it is termed, commensurable. For the fraction $\frac{a}{b}$ means a quantity which when multiplied by b makes a (27). Hence, when a and b are positive integers, $\frac{a}{b}$ is the number of times or parts of times that b has to be multiplied to make a, that is, the number of times or parts of times that b is contained in a. The fraction $\frac{a}{b}$ is also unaltered in value if a and b are both multiplied by or divided by the same factor (101). It therefore agrees as a measure of ratio with the fundamental idea that the ratio of the same two quantities is unaltered whatever the unit be to which their magnitudes are expressed. The weights of two bodies, for instance, have the same ratio whether these weights are both expressed in tons, or in pounds, or in ounces.

The algebraical measure of ratio now given thus includes the arithmetical conception of ratio, and embraces besides incommensurable quantities, or quantities with which we can only deal in arithmetic approximately, by taking their values to a specified number of decimal places.

303. Two magnitudes cannot have a ratio unless the lesser of them can be multiplied so as to make it exceed the greater. Thus, for instance, time can have no ratio to weight or length, because no comparison of greater or less can be made between time and weight, or time and length. This is expressed by terming quantities between which ratio exists as quantities of the same kind, two lengths, for instance, two areas, two durations of time.

Compound Ratio.

304. If there be several magnitudes of the same kind, a_1, a_2, a_3,...a_n, it follows from (101) and (113) that

$$\frac{a_1}{a_n} = \frac{a_1}{a_2} \cdot \frac{a_2}{a_3} \cdot \frac{a_3}{a_4} \cdots \frac{a_{n-1}}{a_n}.$$

In this case a_1 is said to have to a_n the ratio compounded of the ratios of a_1 to a_2, and of a_2 to a_3, and so on. The ratio $a_1 : a_n$ is, we see, the product of the several ratios of which it is said to be compounded.

305. Compounded ratio is exemplified in what is called the Chain Rule in arithmetic, where the value of one quantity is compared with that of another through intermediate links.

Ex. If 3 oxen are worth as much as 10 calves,
2 calves „ „ 3 sheep,
4 sheep „ „ 7 pigs,

to compare the value of an ox and that of a pig.

If a_1 be the value of an ox,
a_2 „ „ calf,
a_3 „ „ sheep,
a_4 „ „ pig.

$$a_1 : a_2 = 10 : 3,$$
$$a_2 : a_3 = 3 : 2,$$
$$a_3 : a_4 = 7 : 4.$$

$$\therefore \frac{a_1}{a_4} = \frac{a_1}{a_2} \cdot \frac{a_2}{a_3} \cdot \frac{a_3}{a_4} = \frac{10}{3} \cdot \frac{3}{2} \cdot \frac{7}{4} = \frac{70}{8} = \frac{35}{4}.$$

Here the value of an ox to that of a pig is a ratio compounded of the ratios of ox to calf, calf to sheep, sheep to pig, and this ratio is found to be $\frac{35}{4}$.

Proportion.

306. *Def.*—Four quantities are in proportion when the ratios of the first to the second, and of the third to the fourth, are equal.

If a, b, c, d, be four quantities where a has a ratio to b, and c has a ratio to d, then the four are in proportion when $\frac{a}{b} = \frac{c}{d}$. The quantities are then sometimes called proportionals.

Obs.—The two quantities meant by a and b in this statement are of the same kind one with the other (303), and c and d are also of the same kind one with the other, though of a different kind, it may be, from a and b.

307. If the four quantities a, b, c, d, are proportionals, when any three of them are known the equation $\frac{a}{b} = \frac{c}{d}$ will determine the fourth.

$$\text{Thus, since } ad = bc,\ d = \frac{bc}{a}.$$

This is an algebraic statement of that which in arithmetic is known as the Rule of Three, whereby a fourth proportional is found to three given magnitudes, the process being that of multiplying together the numbers which express the second and third magnitudes, and dividing the product by the number expressing the first magnitude.

308. *Def.*—Three quantities of the same kind are said to be in proportion when the ratios of the first to the second and of the second to the third are equal.

Thus a, b, c, are in proportion, or proportionals, if $\frac{a}{b} = \frac{b}{c}$. In this case b is called a mean proportional between a and c.

This is nothing but a special case of the preceding more general statement, if the second and third of the four quantities therein considered are supposed to become equal to one another.

If a and b are two given quantities of the same kind, the third proportional to them is $\frac{b^2}{a}$, and the mean proportional is $\sqrt{ab}$.

309. Any even number of magnitudes are called proportionals, or said to be in proportion, when, as they are taken two together in order, the ratios resulting are all equal. Thus $a, b, c, d, e, f, \ldots$ are proportionals if

$$\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \ldots \text{ are all equal fractions.}$$

310. *Def.*—If three quantities are in proportion, the first is said to have to the third the duplicate ratio of that which it has to the second.

If the quantities be a, b, c, so that

$$a : b = b : c, \text{ or } \frac{a}{b} = \frac{b}{c},$$

$$\text{then } \frac{a}{c} = \frac{a}{b} \cdot \frac{b}{c} = \frac{a^2}{b^2}.$$

It appears then that the ratio $\frac{a}{c}$, the duplicate of the ratio $\frac{a}{b}$, according to the definition, is the square of the ratio of a^2 to b^2.

311. *Def.*—In the ratio of a to b, or $\frac{a}{b}$, a and b are the *terms* of the ratio, and the first term a is called the antecedent, and the second term b the consequent.

312. *Def.*—When four quantities, a, b, c, d, are proportionals, or $\frac{a}{b} = \frac{c}{d}$, the antecedents, a and c, in the ratios are said to be homologous to one another, and so are the consequents, b and d.

313. *Problems.*

1. A certain number is added to each term of the ratio $3:10$, and the same number is also subtracted from each term, and it is found that the resulting ratio in the first case is the duplicate of the resulting ratio in the second. What is the number?

Let x be the number. By adding it to each term of the given ratio we have the ratio $\frac{3+x}{10+x}$, and by subtracting it from each term we have the ratio $\frac{3-x}{10-x}$. Now the former is the duplicate (310) of the latter.

$$\therefore \frac{3+x}{10+x} = \left(\frac{-x}{10-x}\right)^2.$$

$$\therefore (10-x)^2(3+x) = (10+x)(3-x)^2,$$
$$\text{whence } x = 6, \text{ or } -\tfrac{5}{3}.$$

If both terms of the ratio under the specified alterations are to continue positive, the latter is the only admissible answer.

2. There are two amalgams of the same bulk, each composed of mercury and gold, in the ratios of $2:9$ and $3:19$, respectively. If they were fused together, what would be the ratio of mercury to gold in the resulting amalgam?

Let there be $11x$ cubic inches in the first,

so that there are $\begin{cases} 2x \text{ of mercury,} \\ 9x \text{ of gold.} \end{cases}$

Let there be $22y$ cubic inches in the second,

so that there are $\begin{cases} 3y \text{ of mercury,} \\ 19y \text{ of gold.} \end{cases}$

Since the amalgams have the same bulk,

$$11x = 22y, \text{ or } x = 2y.$$

Now after fusion, supposing no bulk to be lost, there are

$2x + 3y = 4y + 3y = 7y$ cubic inches of mercury,

and $9x + 19y = 18y + 19y = 37y$ „ „ gold.

$\therefore$ the ratio of the quantities of mercury and gold is $7:37$.

LOGARITHMS.

1. Every positive quantity, integral or decimal, has a number belonging to it called its logarithm. These logarithms are so computed that they fulfil the following four remarkable laws:

1. The logarithms of two numbers added together are the logarithm of the product of these numbers.

2. The logarithm of one number subtracted from the logarithm of another is the logarithm of the quotient resulting from dividing the latter number by the former.

If a, b be two numbers, and $\log a$, $\log b$ mean their logarithms, these laws are thus algebraically expressed:

$$\log a + \log b = \log (ab),$$

$$\log a - \log b = \log \left(\frac{a}{b}\right).$$

3. If the logarithm of a number be multiplied by any integer, the result is the logarithm of the power of the number of which power that integer is the exponent (30).

If the logarithm of a number be divided by any integer, the result is the logarithm of that root of the number which the integer designates (32).

If r be any integer, these two latter laws are thus algebraically expressed:

$$r \log a = \log (a^r),$$

$$\frac{1}{r} \log a = \log \left(a^{\frac{1}{r}}\right).$$

2 No proof can here be offered that numbers must exist possessing the properties under which we call them logarithms, neither can any account be here given of the methods of computing such logarithms. The reader will accept the statement that if such numbers exist, bearing the properties aforesaid, they are called logarithms. He must also accept the tables which are published, recording logarithms for the several numbers to which they profess to belong, though he cannot at present verify the computation of these several logarithms, and he will be informed how he may use these tables to effect with comparative ease many calculations which would otherwise be most laborious.

The truth is, though it requires for its demonstration higher algebra than this book introduces, that not only has every number a logarithm, but it has an infinite variety of logarithms, constructed, as the term is, on different scales or bases. The base of any system of logarithms is defined by the fact that in that system *unity* is its logarithm. The base in ordinary use is 10, and hence with this base $\log 10 = 1$. Logarithms to this base are the only ones which will now be considered in their practical use.

3. Whatever the base be, the logarithm of unity is zero.

For since, by *Art.* 1, $\log a + \log b = \log ab$,

$$\text{let } b = 1 \text{ and } \therefore ab = a,$$

$$\therefore \log a + \log 1 = \log a,$$
$$\therefore \log 1 = 0.$$

4. Confining ourselves, as has been just premised, to 10 as the base, we have by *Art.* 1

$$\log 100 = \log 10^2 = 2 \log 10 = 2,$$
$$\log 1000 = \log 10^3 = 3 \log 10 = 3,$$
$$\log 10000 = 4 \log 10 = 4,$$

and generally, if n be any positive integer,

$$\log 10^n = n.$$

$$\text{Again, } \log \tfrac{1}{10} = \log 1 - \log 10 = -\log 10$$
$$= -1,$$
$$\log \tfrac{1}{100} = \log 1 - \log 100 = -2,$$

and generally, if n be any positive integer,

$$\log \frac{1}{10^n} = -n.$$

5. The same number cannot have two different logarithms to the same base 10. For if it were possible that a had two different logarithms to the base 10, m and n suppose, then

$$\log 10a = \log 10 + \log a = 1 + m,$$
$$\text{also} = 1 + n.$$

$\therefore$ m and n cannot be different.

6. The same logarithm to the base 10 cannot belong to different numbers. For if it were possible that m were the logarithm of different numbers a and b,

$$\log \left(\frac{a}{b}\right) = \log a - \log b = m - m = 0.$$

$$\therefore \frac{a}{b} \text{ must be } 1,$$

or a and b are not different.

Characteristics of Logarithms.

7. It will be observed on inspection of the tables of logarithms to the base 10 that the logarithms of numbers are larger as the numbers are larger. Since log 1 is zero and log 10 is 1, the logarithms of all numbers between 1 and 10 are decimals between 0 and 1. Since log 100 = 2, the logarithms of all numbers between 10 and 100 are unity with a decimal part attached. So the logarithms of all numbers between 100 and 1000 are 2 with a decimal part attached, and generally the logarithm of an integer which has $n+1$ digits is n with a decimal attached.

A logarithm, consisting as it usually does of an integral

and decimal part, is described by calling the integral part its characteristic, and the remaining decimal part its mantissa. Hence the preceeding results are expressed by stating that the characteristic of any number greater than unity is the number of integral digits lessened by 1. Thus 3 is the characteristic of the logarithm of 3847·216, since this number has four integral digits; while 0 is the characteristic of the logarithm of 3·847216, since this number has one integral digit. The logarithm in this last case is wholly decimal.

8. *Obs.*—In a decimal where one or more zeros stand after the decimal point before the rest of the digits, these latter are called the significant digits. Thus, in ·0013, 13 are the significant digits, the two zeros non-significant. So also if an integer, as 51700, concludes with zeros, these are called non-significant.

9. Since the logarithm of 1 is 0, and logarithms decrease with their corresponding numbers, it follows that the logarithms of all numbers wholly decimal are negative. In using these logarithms an important alteration is made in them, for reasons hereafter to be given, whereby the decimal part is always positive, and the characteristic only is negative. Thus if $-3·142657$ be the logarithm of some number, the practice is to alter it to the form $-4+·857343$, and then to write it $\bar{4}·857343$, the negative sign being placed over the digit 4 instead of before it, to express the circumstance that it affects that digit alone, and that the decimal part remains a positive quantity.

With this convention in mind, it will be observed that when the number is a decimal with all its digits significant, in value therefore between 1 and $\frac{1}{10}$, its logarithm is negative, yet not so small as the logarithm of $\frac{1}{10}$ which is -1. Its logarithm therefore will be something between 0 and -1, or -1 with some positive decimal added. Hence -1 is its characteristic.

When the number is a decimal with zero as its first digit, in value therefore below $\frac{1}{10}$ but not so low as $\frac{1}{100}$, its

logarithm is less than -1 but not so small as -2, and so will be -2 with some positive decimal attached. Thus $\bar{2}$ is the characteristic. And generally by following this reasoning it will appear that the characteristic of the logarithm of a number less than unity is negative, and is the number of non-significant digits increased by 1. Thus log. ·00347 has $\bar{3}$ for its characteristic, because there are two non-significant digits in the number.

Ex. The characteristics of the logarithms of

217, 217·35, 2·1756, ·584, ·00037,

are respectively 2, 2, 0, $\bar{1}$, $\bar{4}$.

10. Conversely, if we know the characteristic of a logarithm, we know the number of digits in the number and limits between which its value lies. If 5 is the characteristic, the number is one of 6 integral digits, and between 100000 and 999999. If $\bar{2}$ is the characteristic, the number is less than $\frac{1}{10}$ but not less than $\frac{1}{100}$.

11. As long as numbers have the same digits in the same order, whatever be the local value of those digits, the logarithms of such numbers have the same decimal part, the characteristics only being different.

For instance,
$$375 = 100 \times 3{\cdot}75,$$
$$\therefore \log 375 = \log 100 + \log 3{\cdot}75,$$
$$= 2 + \log 3{\cdot}75,$$

so that the logarithm of 3·75 is converted into the logarithm of 375 by adding 2 to the characteristic, the decimal part remaining the same.

$$\text{So } \log {\cdot}00375 = \log \frac{3{\cdot}75}{1000},$$
$$= \log 3{\cdot}75 - \log 1000,$$
$$= \log 3{\cdot}75 - 3,$$
$$\text{and } \log 37500 = \log 3{\cdot}75 \times 10000,$$
$$= \log 3{\cdot}75 + 4.$$

Thus, whatever the number be in which 375 are the

significant digits, whether zeros be attached to the right, or the decimal point be made to take any proposed position, the logarithm of every number thus produced is known if we only know the logarithm of 3·75.

To express this principle generally in algebraical symbols,

$$\text{if } a = 10^n b, \ n \text{ being any integer,}$$

$$\log a = n + \log b.$$

So that if there be two numbers, of which one results from multiplying or dividing the other by 10 or any power of 10, their logarithms have the same decimal part, the characteristics only being different.

This is the reason why the decimal parts of the logarithms of numbers less than unity are made positive (9). It is in order that these logarithms may have the same decimal parts as the logarithms of numbers greater than unity with the same significant digits.

Logarithmic Tables.

12. If the reader will open an ordinary table of the logarithms of numbers, he will find a vertical column on the left side of the page containing four digits, and ten columns of logarithms headed by the digits 0, 1, 2 . . . 9. These last are fifth digits to be attached to the former four, so that the table thus embraces numbers from 10,000 up to 99,999. Opposite to every such number is a number with seven places of figures. This is a decimal, though to save printing the decimal point is not printed, and it is the decimal part of the logarithm of the number to which it corresponds. It will be observed that to save space the first three digits of the logarithm are not printed over and over again, but are printed once for all in the column headed by 0, to be prefixed to the four remaining figures of the following records. The characteristics are never printed, but are to be prefixed according to the rules of (7) and (9).

Hence from such a table we can take out the logarithm of any number with any five significant digits.

For instance, if 69754 be the number, we find in the interval horizontally on a level with 6975 and vertically under 4, the figures 5691. These are the last four figures of the logarithm, the former three being 843 printed only in the first column on the left. Thus the decimal part of the logarithm required is ·8435691, and since the number has five integral digits, the characteristic of the logarithm is 4, and we have

$$\log 69754 = 4{\cdot}8435691.$$

Since the decimal part is the same whatever be the local value of the digits, and the characteristic alone is altered,

$$\log 69754000 = 7{\cdot}8435691,$$
$$\log 697{\cdot}54 = 2{\cdot}8435691,$$
$$\log {\cdot}0069754 = \bar{3}{\cdot}8435691,$$

and so on.

13. *Caution.*—In some parts of the tables it will be observed that a horizontal bar is printed over the fourth digit of a logarithm. The meaning of this is that for that logarithm, and those which follow in its horizontal line, the third digit is to be increased by unity. For instance, in the horizontal line of logarithms against 6934, we have such a bar printed in the fourth vertical column. It is thus expressed that whereas the decimal parts of the logarithms of

69340		·8409838
69341	are	·8409901
69342		·8409964,

the decimal parts of the logarithms of

69343		·8410026
69344		·8410089
69345		·8410152
69346	are	·8410214
69347		·8410277
69348		·8410339
69349		·8410402.

The purpose of this contrivance is that the first three digits of the decimal part shall be printed only once for all the ten logarithms in the same horizontal line.

Mr. Babbage in his Table of Logarithms prefers to print the fourth digit about half the usual size where the third digit has to be increased by unity, and not to use the horizontal bar which was adopted by Hutton.

14. Although, as has been already stated, it requires Algebra beyond the limits of this book to show how logarithms are computed, and thus to enable the student to verify for himself the records of a published table, yet he may now satisfy himself by some instances that the numbers recorded as logarithms are logarithms according to the definition of (1).

By reference to the tables, log 2, having the same decimal part as log 20000, with 0 as characteristic (7), is therefore ·3010300, and log 3 for a similar reason is ·4771213. These logarithms added together make ·7781513, which is the logarithm of 6 in the tables; so that the logarithms of 2 and 3 added together are the logarithm of 6, the product of 2 and 3.

$$\begin{aligned} \text{So } \log 1{\cdot}5 &= \phantom{\bar{1}}{\cdot}1760913 \\ \log {\cdot}4 &= \bar{1}{\cdot}6020600 \\ \hline \text{the sum} &= \bar{1}{\cdot}7781513, \end{aligned}$$

and the corresponding logarithm in the tables is that of ·6, so that the logarithms of 1·5 and ·4 added together make the logarithm of ·6 which is the product of 1·5 and ·4.

In these instances the law (1) of (1) is fulfilled, and now to exemplify law (2).

$$\begin{aligned} \log 9600 &= 3{\cdot}9822712 \\ \log 1{\cdot}2 &= {\cdot}0791812 \\ \hline \text{difference} &= 3{\cdot}9030900, \end{aligned}$$

which from the tables is the logarithm of 8000. Thus the

logarithm of 1·2 subtracted from that of 9600 gives the logarithm of the number resulting from dividing 9600 by 1·2.

Again, to exemplify law (3),

$$\log 200 = 2\cdot30103 \text{ from the tables.}$$

If this logarithm be multiplied by 3, the result 6·90309 is from the tables the logarithm of 8000000, the cube of 200.

$$\text{Also } \log 160000 = 5\cdot2041200.$$

If this logarithm be divided by 4, we have

$$1\cdot3010300$$

which the tables give as the logarithm of 20, and 20 is known to be the fourth root of 160000.

15. Suppose, however, that to exemplify law (1) in the instance of adding the logarithms of 15 and 3, the product of which two numbers is 45, we had taken from the tables

$$\begin{aligned} \log 15 &= 1\cdot1760913 \\ \log\ \ 3 &= \underline{\ \ \cdot4771213} \\ \text{the sum} &= 1\cdot6532126, \end{aligned}$$

and this sum is not according to the tables the logarithm of 45, that logarithm being 1·6532125. A fundamental character of logarithms therefore in this instance, as it appears, fails to be fulfilled.

The explanation is this: The logarithms in the tables are not in general exact, but represent, as far as the seventh place, a decimal extending much further. Few numbers have their logarithms expressed by a terminating decimal. The logarithms of the rest are given to the seventh decimal place. Thus:

$$\begin{aligned} \text{the logarithm of 15 is } & 1\cdot176091259055\ \ldots \\ \text{and that of 3 is } & \underline{\ \ \cdot477121254719\ \ldots} \\ \text{their sum} = & 1\cdot65321251377\ \ \ldots \\ \text{and the logarithm of 45 is } & 1\cdot65321251377\ \ \ldots \end{aligned}$$

so that the agreement is perfect of the logarithms of 15 and 3

making together that of 45. In the tables, however, limited to seven places, the last digit of log 15 was made 3, with the error ·000000041 . . . in excess, and the last digit of log 3 was made 3 with an error ·000000045 . . . in excess also, and the addition of these logarithms consequently appears with an error in excess ·000000086 . . . which represented to the seventh place is ·0000001.

16. This instance may raise the question whether calculations made by logarithms which are taken to a limited number of places of decimals are exact and trustworthy, and the answer is, that in few instances are these calculations exact, but the results obtained are so very near exactness that they are practically of the same use to us as if they were, in cases where logarithms are commonly employed. Thus, to revert to the example just considered, 1·6532126 is not, according to the tables, the logarithm of 45, but if by methods presently to be given we proceeded to find the number of which it is the logarithm, that number would prove to be 45·10526.... Since logarithms are generally used where the numbers introduced have many places of figures, an error affecting the last place only of the result is of less consequence in practice. It must not be disguised, however, that logarithms are a method of approximate computation, and are not sure to give results which are rigidly accurate.

Proportional Parts.

17. It has already been seen (12) how the tables and the law of characteristics enable us at once to write down the logarithm of any number of five digits, whatever be the local value of these digits. The case is now to be considered of the logarithms of numbers which have more than five significant digits, and we have to make the tables available for finding these logarithms.

Since the decimal part of the logarithm, which alone there is any difficulty in determining, has no respect to the posi-

tion of the decimal point in the number, suppose that the number whose logarithm is required is 54153·4. Now if the page of logarithms be examined wherein are given the decimal parts for the numbers about 54153, it will be observed that for some extent on each side of 54153 an increase of unity in the number has the effect of increasing the logarithm by ·000008. Since the logarithms, therefore, about this part of the tables are growing uniformly as the number increases by successive units, it is assumed that the law of uniform growth will apply to alterations of the number less than unity, and that an increase ·4 of the number 54153 will add to the logarithm $\frac{4}{10}$ of the addition whereby we step to the logarithm of 54154. Hence we find the logarithm of 54153·4 if to the logarithm of 54153 we add $\frac{4}{10}$ of the difference ·000008, or ·0000032.

$$\begin{aligned} \text{Thus } \log 54153 &= 4{\cdot}73336225 \\ &\quad\ \ \underline{\;{\cdot}0000032} \\ \log 54153{\cdot}4 &= 4{\cdot}73336257. \end{aligned}$$

From this, by alteration of characteristics, come at once the logarithms of 54·1534, ·0541534, and of all numbers with these same digits, whatever be the local value of the digits.

18. The principle here explained is of extensive use with all tables wherein results are recorded at fixed intervals, and it enables us to obtain the proper record for other given points in the course of such intervals. The assumption is, that the quantity recorded is at the part of the table under consideration in a state of steady growth. The proportionate addition made to the record is called the proportional part, and the principle employed is called the method of proportional parts.

The determination of the logarithm of a number of more than six significant digits is now only an extension of the method just given. For instance, if we were wanting the logarithm of 54153·496, we should have to find the pro-

portionate addition for ·496 by taking ·496 of ·000008 or ·000003968, which to the seventh place of decimals is ·000004,

$$\therefore \log 54153 = 4{\cdot}7336225,$$
$$\text{proportional part for } {\cdot}496 = {\cdot}000004,$$
$$\therefore \log 54153{\cdot}496 = 4{\cdot}7336265.$$

19. Ex. Given $\log 6{\cdot}1025 = {\cdot}7855078$,
$\log 6{\cdot}1026 = {\cdot}7855149$.

find log 610·257.

Science Examination 1869.

From the given logarithms

$$\log 610{\cdot}26 = 2{\cdot}7855149 \ (11),$$
$$\log 610{\cdot}25 = 2{\cdot}7855078,$$
$$\text{difference} = {\cdot}0000071.$$

Thus an increase of ·01 in the number produces an increase of ·0000071 in the logarithm. It is to be ascertained what, at the same assumed rate, will be the increase of the logarithm consequent on an increase of ·007 in the number.

If an increase ·01 in the number gives an increase ·0000071 in the logarithm.

∴ an increase ·001 in the number gives an increase ·00000071 in the logarithm.

∴ an increase ·007 in the number gives an increase ·00000497 in the logarithm.

$$\begin{aligned} \text{Hence } \log 610{\cdot}257 &= 2{\cdot}7855078 \\ &\quad\ \underline{{\cdot}00000497} \\ &\quad\ 2{\cdot}78551277 \\ &= 2{\cdot}7855128 \end{aligned}$$

to seven places of decimals.

20. To relieve computers from the trouble of calculating the proportional addition when more than five significant digits are in a number whose logarithm they require, the proportional parts are printed at the foot or the side of the page of logarithms, in what is called a table of proportional parts, as they correspond to each additional digit. Thus, in the

part of the table from which an example has just been taken, there is a table at the side thus printed:

D	P								
80	8	16	24	32	40	48	56	64	72

and the meaning is that when ·0000080 is the difference between two consecutive logarithms in the table, if there is an additional sixth digit in the number, then when the logarithm belonging to the five digits has been taken out, there must be added to this logarithm

·0000008 when the sixth digit is 1,
·0000016 „ „ 2,
·0000024 „ „ 3,

and so on.

When there is a seventh or eighth digit in the number, the additional proportional part in the tables is to be divided by 10 or 100.

A table of proportional parts is to be used throughout that extent of the table of logarithms wherein the logarithms advance by the prefixed difference. In some pages of the table one table of proportional parts suffices. In other pages the differences vary, and two or more tables of proportional parts have accordingly to be constructed.

The proportional parts are computed as decimals taken to the seventh place. Suppose at a part of the tables the difference is ·0000078. The proportional parts for the digits 1, 2, 3,... being $\frac{1}{10}$, $\frac{2}{10}$, $\frac{3}{10}$,... of this difference are consequently

$$
\begin{aligned}
\cdot 00000078 &= \cdot 0000008, \\
\cdot 00000156 &= \cdot 0000016, \\
\cdot 00000234 &= \cdot 0000023, \\
\cdot 00000312 &= \cdot 0000031, \\
\cdot 0000039 &= \cdot 0000039, \\
\cdot 00000468 &= \cdot 0000047, \\
\cdot 00000546 &= \cdot 0000055, \\
\cdot 00000624 &= \cdot 0000062, \\
\cdot 00000702 &= \cdot 0000070,
\end{aligned}
$$

and the table therefore records as proportional parts

8 | 16 | 23 | 31 | 39 | 47 | 55 | 62 | 70.

21. It will be observed that at the earlier part of a table of logarithms the differences are larger in magnitude, and also change more frequently. As we advance in the table the differences grow less in magnitude, and the same difference applies to a larger number of recorded logarithms.

22. The reader is now qualified to find, by means of the tables, the logarithm of any number whatever, the logarithm being correct to the seventh place of decimals, to which number of places it is supposed that the tables are constructed. He will proceed according to the following method.

Write down the characteristic by the rules of (7) and (9).

If the number proposed has not more than five places of significant figures, write down its decimal part taken out of the tables, and the logarithm required is completed.

But if the number has more than five places of significant figures, write down out of the tables the decimal part of the first five digits, add from the table of proportional parts the addition for the sixth digit, then the further addition for the seventh digit, and by summing up the whole the complete logarithm is found.

23. *Obs.*—In operations of any length, wherein logarithms are used, it is a convenient practice, in order to secure the digits ranging one under another in their proper positions, to rule a vertical line down the paper and write on the right side of this the last four digits of the logarithm, and the rest of it on the left side. The last four digits of the logarithms are conveniently written down at the first inspection of the tables and then the remaining three.

24. Ex. 1. To find the logarithm of ·00594.

The tables at once give $\bar{3}\cdot7737864$.

Ex. 2. To find the logarithm of ·3461171.

$$\begin{array}{lrl} \log \cdot 34611 = & \bar{1}\cdot 539 & 2141 \\ \text{prop. part for } 7 & & 88 \\ \text{,, \quad ,, } 1 & & 1 \\ \hline & \bar{1}\cdot 539 & 2230 \end{array}$$

∴ log ·3461171 is $\bar{1}$·539223.

Ex. 3. To find the logarithm of 295·8536.

From the tables, after the characteristic 2 is prefixed,

$$\begin{array}{rrl} \log 295\cdot 85 = & 2\cdot 471 & 0716 \quad (13) \\ \text{prop. part for } 3 = & & 44 \\ \text{,, \quad ,, } 6 = & & 9 \\ \hline \log 295\cdot 8536 = & 2\cdot 471 & 0769. \end{array}$$

Obs.—9 is taken as the proportional part for the seventh digit 6, because ·00000088 is represented by ·0000009 to seven places of decimals.

25. *Examples for Practice.*

1. log 57·0828 = 1·7565055
2. log 2137775 = 6·3299621
3. log 2·901793 = ·4626665
4. log ·0019777677 = $\bar{3}$·2961553.

26. The converse process is now to be described.

When a logarithm is presented, to find the number of which it is the logarithm.

If the decimal part of the logarithm is found in the tables, the corresponding number is at once known, the number of its integral places being regulated by the characteristic of the given logarithm.

Ex.	2·7500839	is the logarithm	of	562·45
	5·7480717	,,	,,	559850
	$\bar{3}$·7490248	,,	,,	·056108 (13).

27. If the decimal part of the logarithm presented is not found in the tables, but lies between two successive records in the tables, the number is to be found by the method of proportional parts, on the supposition that between two successive records in the tables the number advances in proportion to the increase of the logarithm.

Suppose the logarithm presented is 2·7480893. It is larger than the logarithm of 559·87, and less than the logarithm of 559·88. The difference of these two logarithms is ·0000078. Since, then, as the logarithm receives this increase, the number has advanced by ·01, a proportion is to give the advance of the number for an increase ·0000015 in the logarithm.

·0000078, increase of the log, increases the number by ·01

$$\cdot 0000015 \quad " \quad " \quad " \quad \frac{\cdot 000000015}{\cdot 0000078}$$

or ·002.

∴ the number corresponding to the given logarithm is 559·872.

28. As in finding a logarithm, so also in finding the number indicated by a logarithm, the table of proportional parts dispenses with the necessity of computing the proportional addition. The following will exemplify the use of it:

Required the number belonging to the logarithm ·4626665.

The next lower record in the tables gives

$$\begin{array}{lr} \log 2\cdot 9017 = & \underline{\cdot 4626525} \\ \text{excess} & 140 \end{array}$$

Now from the table of proportional parts for this part of the table, where ·0000150 is the difference of two successive records,

$$\begin{array}{lr} \text{prop. part for } 9 = & \underline{135} \\ \text{leaving difference} & 5 \end{array}$$

the proportional part which, when divided by 10, comes

nearest to this is 45, the proportional part for 3, and thus the number so far determined is pronounced to be 2·901793.

29. The reader can exercise himself in the following examples, either by finding the logarithms of the numbers given, or by finding the numbers, supposing the logarithms given.

1.	log ·0724658	=	$\bar{2}$·8601331.
2.	log ·0086598642	=	$\bar{3}$·9575111.
3.	log 19743267	=	7·2954190.
4.	log 887·17777	=	2·9480106.
5.	log ·0394068	=	$\bar{2}$·5955712.
6.	log ·5688975	=	$\bar{1}$·7750340.
7.	log 158·8888	=	2·2010933.
8.	log 6·622894	=	·8210478.
9.	log ·05319876	=	$\bar{2}$·7259015.
10.	log 3·37307	=	·52017856.

30. The method being understood for finding

(1) the logarithm of any proposed number,
(2) the number belonging to any proposed logarithm,

the reader is now qualified to use logarithms to perform the multiplication, division, involution and evolution of numbers, and to appreciate the power they give of effecting computations which would otherwise be almost unapproachable from their complication and tediousness.

Multiplication by Logarithms.

31. By the definition of logarithms, when the logarithms of two numbers are added together, we have the logarithm of their product, and the product itself can then be determined (26).

Ex. To multiply 598·6437 by ·05462398.

log 598·6437 =	log 598·64	2·777 1657
	+ prop. part for 3	22
	+ " " 7	5
log ·05462938 =	log ·054629	$\bar{2}$·737 4233
	+ prop. part for 3	24
	+ " " 8	6
	log of product	1·514 5947
	Now log 32·703 =	1·514 5876
		71
	prop. part for 5	67
		4
	" " 3	4

∴ the product is 32·70353.

Since the numbers end in 7 and 8, the last digit of their product is 6, and the result obtained is not exact to the last digit. This is another instance to exemplify what was explained in (16) that logarithms taken to seven places are not in general exact values, and the results of working with them may not be trustworthy to the last place of figures. For the purposes where logarithms are employed such a departure from exactness is not practically inconvenient.

32. By extension of this method the logarithms of several numbers more than two give, when added together, the logarithm of their product.

For $$\log (abc) = \log (ab) + \log (c)$$
$$= \log a + \log b + \log c,$$

and so if more factors appear.

Ex. To find the product of 8470, ·053917, 48·306, ·48309.

It will be found in the tables that

$$
\begin{array}{rcl}
\log 8470 &=& 3\cdot927\,8834 \\
\log \cdot053917 &=& \bar{2}\cdot731\,7257 \\
\log 48\cdot306 &=& 1\cdot684\,0011\ (13) \\
\log \cdot48309 &=& \bar{1}\cdot684\,0280 \\
\therefore \log \text{ of the product} &=& 4\cdot027\,6382 \\
\text{Now } \log 10657 &=& 4\cdot027\,6350 \\
&& \qquad 320 \\
\text{prop. part for } 07 && \qquad 285 \\
&& \qquad\ \ 350 \\
\text{,,} \quad \text{,,} \quad 9 && \qquad\ \ 366
\end{array}
$$

$\therefore$ the product so far determined is 10657·079.

33. *Examples for Practice.*

1. $13\cdot07564 \times 13\cdot7564 = 179\cdot8738$.
2. $\cdot5684325 \times 893 = 507\cdot6$.
3. $47\cdot3943 \times \cdot68974 = 32\cdot68981$.
4. $167\cdot75384 \times \cdot656247 = 110\cdot088$.
5. $\cdot647853 \times \cdot0712384 \times 1\cdot359865 = \cdot0627605$.

Division by Logarithms.

34. If one number, the dividend, is to be divided by any other, the divisor, the logarithm of the result or quotient is found by subtracting the logarithm of the divisor from that of the dividend, and then the quotient itself can be found from its logarithm.

The only difficulty which division will raise is when the logarithm to be subtracted is greater than the logarithm from which it has to be subtracted, and when negative characteristics appear. The computer has to keep in mind in such cases the meaning of the quantities with which he is operating, that the decimal part of a logarithm is always positive, while its characteristic may be either positive or negative.

For instance, to divide 30·485 by ·052525.

From log 30·485 =	1·484 0862
is to be subtracted	$\bar{2}$·720 3661
and the result is	2·763 7201
Now log 580·39 =	2·763 7199
	2
prop. part for 03	2

∴ the quotient is 580·3903.

It is to be remembered that the logarithm to be subtracted is $-2 + \cdot 7203661$, or the negative number $-1\cdot 2796339$.

35. When the divisor has more than 5 places of significant figures, so that its logarithm cannot be written down by one reference to the tables, but has to be formed by the addition of a proportional part, it is a convenient arrangement to draw a distinct double line under the logarithm of the dividend, and when the logarithm of the divisor is formed which is to be subtracted from the former logarithm, the eye is readily carried up to the proper line.

Ex. To find the result of dividing 186·329 by ·06842973.

log 186·32 =	2·270 2595	
addition for 9	210	
log 186·329 =	2·270 2805	(*a*)
log ·068429 =	$\bar{2}$·835 2402	
addition for 7	44	
„ „ 3	2	
log ·06842973 =	$\bar{2}$·835 2448	(*b*)
∴ log of quotient =	3·435 0357	
log 2722·9 =	3·435 0317	
	40	
prop. part for 2	32	
	8	
„ „ 5	8	

whence the quotient is 2722·925.

The use of the double line enables the eye in subtracting the logarithm (b) readily to catch the successive figures of (a), from which the subtraction is made, and allows the numerical operation to stand in one column.

[This arrangement, with other valuable suggestions, was brought to my notice by the Principal of the Normal School of the Royal Military Asylum, W. G. Reynolds, Esq., whose success in teaching the use of logarithms for practical computations gives value to his recommendations.—W. N. G.]

Arithmetic Complement.

36. To make the process of division less open to error from mistakes, when logarithms with negative characteristics would be subtracted, the arithmetic complement of a logarithm is used. The arithmetic complement of a logarithm is the logarithm subtracted from 10. It can be written down at sight by subtracting each digit of the logarithm from 9, except the last significant digit, and subtracting that from 10.

Thus, the arithmetic complements of

$$\begin{array}{lcl} 1{\cdot}3987654 & & 8{\cdot}6012346 \\ \bar{2}{\cdot}4209876 & \text{are} & 11{\cdot}5790124 \\ 12{\cdot}5980437 & & \bar{3}{\cdot}4019563. \end{array}$$

Now, if a is to be divided by b,

$$\begin{aligned} \log\frac{a}{b} &= \log a - \log b \\ &= \log a + 10 - \log b - 10 \\ &= \log a + \text{arithmetic complement of} \log b - 10. \end{aligned}$$

Hereby the rule for division may thus be modified.

Add together the logarithm of the dividend and the arithmetic complement of the logarithm of the divisor, subtract 10 from the sum, and the logarithm of the quotient is found.

The purpose for which the arithmetic complement is used is, that the subtraction of one logarithm from another shall be required. The student is recommended to exercise himself in performing division by both methods, with and

without the arithmetic complement, and practice will show him which method he will find the more easy and certain in any case which presents itself.

Ex. To divide 32·906 by ·004397623.

log 32·906	1·517 2751 (*a*)
log ·0043976	$\bar{3}$·643 2157
addition for 2	20
" " 3	3
∴ log ·004397623	$\bar{3}$·643 2180
of which the arithmetic complement is	12·356 7820 (*b*)
(*a*) and (*b*) together give	13·874 0571
	10
∴ logarithm of quotient	3·874 0571
now log 7482·6	3·874 0525
	46
prop. part for 7	41
	5
" " 9	5

∴ the quotient required is 7482·679.

37. *Examples of Division for Practice.*

1. $\dfrac{8·64}{225·724}$ = ·03827654.

2. $\dfrac{864·3}{·0849362}$ = 10175·87.

3. $\dfrac{1}{21979}$ to 7 places of decimals = ·0000455.

4. $\dfrac{320}{3·207693}$ = 99·7602 to four places of decimals.

5. $\dfrac{·034}{864·527}$ = ·000039328.

6. $\dfrac{·507463}{·0507925}$ = 9·99.

Involution by Logarithms.

38. Involution presents no new difficulty. The logarithm of the number to be raised to a power is to be multiplied by the exponent of the power.

To raise ·0836 to the fifth power.

$$\begin{array}{rl} \log \cdot 0836 = & \bar{2}\cdot 922\,2063 \\ & \underline{5} \\ & \bar{6}\cdot 611\,0315 \end{array}$$

Obs.—The result of multiplying the decimal part, which is positive, by 5 is 4·6110315. To this is to be attached the product of -2 by 5, which is -10, and the result gives $\bar{6}$ as the characteristic.

$$\begin{array}{lr} \text{Now log } \cdot 0000040834 = & \underline{\bar{6}\cdot 6110219} \\ & 96 \\ \text{prop. part for } 9 & \underline{96} \end{array}$$

∴ the power is ·00000408349.

39. To find the whole number whose fifth root is nearest to 18·5.

If N represents the number whose fifth root is 18·5.

$$N = (18\cdot 5)^5,$$

$$\therefore \log N = 5 \log 18\cdot 5,$$

$$\begin{array}{lr} & = 6\cdot 335\,8585 \\ \log 2166900 = & 6\cdot 335\,8389 \\ & 196 \\ \text{prop. part for } 9 & \underline{180} \\ & 160 \\ \text{"} \quad \text{"} \quad 8 & \underline{160} \end{array}$$

∴ $N = 2166998$.

Although this result appears to be the exact fifth power of 18·5, yet it is only this so far as the tables of logarithms represent the logarithms. It is evident that the exact fifth power of 18·5 must have 5 for its final digit, and cannot be a whole number. The result obtained, therefore, according to the terms of the question, is the nearest integer to the complete fifth power, as far as the tables of logarithms are to be trusted.

40. *Examples for Practice.*

1. $(\cdot540968)^5 = \cdot0463297.$
2. $(\cdot545124)^5 = \cdot0481367.$
3. $(4\cdot8755)^7 = 68483\cdot86.$ (13)
4. $\dfrac{(34\cdot762)^2}{\cdot0374} = 32310.$
5. $\dfrac{(\cdot0392748)^7}{\cdot00093624} = \cdot00000015396.$
6. When x is 5, $(\frac{5}{3})^x = 12\cdot8.$

Evolution by Logarithms.

41. Evolution is effected by dividing the logarithm of the number whose root is required by the number which designates the root. It is only to be remembered that where the characteristic is negative the decimal part is still positive, and the logarithm requires a little alteration of form in order to divide it conveniently. If $\bar{3}\cdot1459684$ were to be divided by 4, the number being really $-2\cdot8540316$, the quotient is $-\cdot7135079$, or in the usual form of a logarithm $\bar{1}\cdot2864921$. This is conveniently obtained at once by placing $\bar{3}\cdot1459684$ under the form $-4+1\cdot1459684$, whereby the quotient is at once seen to be $-1+\cdot2864921$, or $\bar{1}\cdot2864921$ in the usual manner of writing logarithms. So if the logarithm $\bar{5}\cdot2189764$ were to be divided by 3, it would be placed under the form $\bar{6}+1\cdot2189765$, and give the result $\bar{2}\cdot4063255$.

42. Ex. 1. To extract the fifth root of 7.

$$\begin{aligned} \log 7 &= \cdot 8450980 \\ \log 7^{\frac{1}{5}} &= \cdot 1690196 \\ \log 1\cdot 4757 &= \cdot 1689981 \end{aligned}$$

$$\begin{array}{lr} & 215 \\ \text{prop. part for } 7 & 206 \\ \hline & 9 \\ \text{,, \quad ,, \quad } 3 & 8\cdot 8 \end{array}$$

$\therefore\ 7^{\frac{1}{5}} = 1\cdot 475773.$

Ex. 2. To extract the fourth root of $\cdot 003624937$.

$$\begin{array}{rl} \log \cdot 0036249 = & \bar{3}\cdot 5592960 \\ \text{prop. part for } 3 & 36 \\ \text{,, \quad ,, \quad } 7 & 8 \\ \hline \log \cdot 003624937 = & \bar{3}\cdot 5593004. \end{array}$$

$$\begin{aligned} \therefore\ \log(\cdot 003624937)^{\frac{1}{4}} &= \bar{1}\cdot 3898251. \\ \text{Now } \log \cdot 24537 &= \bar{1}\cdot 3898215 \end{aligned}$$

$$\begin{array}{lr} & 36 \\ \text{prop. part for } 2 & 35 \end{array}$$

$\therefore\ (\cdot 003624937)^{\frac{1}{4}} = \cdot 245372.$

Ex. 3. Extract by logarithms the seventh root of $\cdot 0047681$.

Science Examination 1864.

$$\begin{aligned} \log \cdot 0047681 &= \bar{3}\cdot 6783454 \\ &= -7+4\cdot 6783454 \\ \therefore\ \log \sqrt[7]{\cdot 0047681} &= \bar{1}\cdot 6683351 \\ \log \cdot 46594 &= \bar{1}\cdot 6683300 \end{aligned}$$

$$\begin{array}{lr} & 51 \\ \text{prop. part for } 5 & 47 \\ \hline & 40 \\ \text{,, \quad ,, \quad } 4 & 37 \end{array}$$

$\therefore\ \sqrt[7]{\cdot 0047681} = \cdot 4659454.$

43. *Examples for Practice.*

1. $\sqrt[10]{\cdot 34} = \cdot 89773$.
2. $\sqrt[5]{\cdot 03} = \cdot 495935$ to six places of decimals.
3. $\sqrt[5]{\cdot 034} = \cdot 508$.
4. $\sqrt[7]{\cdot 03990873} = \cdot 631179$.
5. $\sqrt[12]{\cdot 04568873} = \cdot 7732436$.
6. $\dfrac{\sqrt[3]{37\cdot 2}}{38\cdot 41513} = \cdot 08689843$.
7. $\dfrac{\sqrt{\cdot 0374}}{3741} = \cdot 000051695$.
8. $\dfrac{\sqrt[3]{\cdot 14623}}{82\cdot 63947}$ to six places of decimals $= \cdot 006376$.
9. $\sqrt[4]{\cdot 086} \times 39\cdot 86427 = 21\cdot 5878$.
10. $\sqrt{\dfrac{307 \times 963}{856 \times 1042}} = \cdot 5756$.
11. $\sqrt[5]{\dfrac{17\cdot 758 \times 384\cdot 926 \times 243\cdot 847}{25\cdot 9683 \times 475\cdot 79 \times \cdot 157296}} = 3\cdot 860685$.

44. The student has now the principles before him on which, by aid of a table of logarithms, he can compute the value of any numerical expression of one term, since such an expression will be formed by no operations but multiplication, division, involution and evolution. Besides examples formed from numbers taken at random, some expressions shall now be computed which arise in practice.

45. If there be a solid wheel whose thickness is 4 inches, and its circumference $7\cdot 27$ inches, it will have for its content $\dfrac{(7\cdot 27)^2}{3\cdot 14159}$ cubic inches. [For this statement the reader is referred to books on Mensuration.] To compute this expression by logarithms :

$$
\begin{array}{lrl}
\log 7{\cdot}27 = & {\cdot}861 & 5344 \\
 & & 2 \\
\hline
 & 1{\cdot}723 & 0688 \\
\log 3{\cdot}1415 = & {\cdot}497 & 1371 \\
\text{prop. part for } 9 & & 124 \\
\hline
 & {\cdot}497 & 1495 \\
\hline
 & 1{\cdot}225 & 9193 \\
\log 16{\cdot}823 = & 1{\cdot}225 & 9034 \\
\hline
 & & 159 \\
\text{prop. part for } 5 & & 130 \\
\hline
 & & 29 \\
\text{,, ,, } 9 & & 23
\end{array}
$$

∴ content required is 16·82359 cubic inches.

An instance like this will show that though logarithms are not instruments of exact computation (16), yet the results which they obtain are true as closely as in practice we require them to be. Suppose, for instance, that the last two figures of the result just obtained are not exact, the error does not amount to the thousandth part of a cubic inch.

46. If the circumference of a spherical ball be measured and found to be 14·26 inches, it contains $\cdot 016887 \times (14{\cdot}26)^3$ cubic inches. [See books on Mensuration.]

To compute this expression :

$$
\begin{array}{lrl}
\log 14{\cdot}26 = & 1{\cdot}154 & 1195 \\
 & & 3 \\
\hline
\log (14{\cdot}26)^3 = & 3{\cdot}462 & 3585 \\
\log {\cdot}016887 = & \bar{2}{\cdot}227 & 5525 \\
\hline
\text{sum} = & 1{\cdot}689 & 9110 \\
\log 48{\cdot}967 = & 1{\cdot}689 & 9035 \\
\hline
 & & 75 \\
\text{prop. part for } 9 & & 79
\end{array}
$$

∴ the content of the ball is 48·9679 cubic inches.

47. *Caution.*—The logarithms of numbers do not by addition make the logarithm of the sum of those numbers.

To compute the value of $\{\sqrt{3 \cdot 0479}+\sqrt[3]{3479}\}^3$.

In this example it will be our method first to find by separate operations the values of $\sqrt{3 \cdot 0479}$ and of $\sqrt[3]{3479}$. These values being added together, their sum is then, by aid of logarithms, to be raised to the third power.

$$
\begin{aligned}
\log 3 \cdot 0479 &= \cdot 484\,0007 \quad (13) \\
\log \sqrt{3 \cdot 0479} &= \cdot 242\,0004 \\
\text{Now } \log 1 \cdot 7458 &= \cdot 241\,9945 \\
& \qquad\quad 59 \\
\text{prop. part for } 2 & \qquad\quad 50 \\
\therefore \sqrt{3 \cdot 0479} &= 1 \cdot 74582.
\end{aligned}
$$

$$
\begin{aligned}
\text{Again, } \log 3479 &= 3 \cdot 541\,4544 \\
\log \sqrt[3]{3479} &= 1 \cdot 180\,4848 \\
\log 15 \cdot 152 &= 1 \cdot 180\,4700 \\
& \qquad\quad 148 \\
\text{prop. part for } 5 & \qquad\quad 143 \\
\therefore \sqrt[3]{3479} &= 15 \cdot 1525.
\end{aligned}
$$

$$
\begin{aligned}
\therefore \sqrt{3 \cdot 0479}+\sqrt[3]{3479} &= 16 \cdot 89832. \\
\log 16 \cdot 898 &= 1 \cdot 227\,8353 \\
\text{prop. part for } 3 & \qquad\quad 77 \\
\text{,, \quad ,, \quad } 2 & \qquad\quad 5 \\
\therefore \log. 16 \cdot 89832 &= 1 \cdot 227\,8435 \\
\log. (16 \cdot 89832)^3 &= 3 \cdot 683\,5305 \\
\log. 4825 \cdot 3 &= 3 \cdot 683\,5243 \\
& \qquad\quad 62 \\
\text{prop. part for } 7 & \qquad\quad 63
\end{aligned}
$$

$\therefore$ the result required is $4825 \cdot 37$.

48. *Sundry Examples.*

Ex. 1. If log 2 = ·30103, find, without the tables, the logarithm of 250. *Science Examination* 1866.

$$250 = \frac{1000}{4} = \frac{1000}{2^2}.$$

$$\begin{aligned} \therefore \log. 250 &= \log 1000 - 2 \log 2 \\ &= 3 - {\cdot}60206 \\ &= 2{\cdot}39794. \end{aligned}$$

Ex. 2. The logarithm of 2 is ·30103, find, without the tables, the logarithm of ·625.

$${\cdot}625 = \frac{10}{2^4}.$$

$$\begin{aligned} \therefore \log {\cdot}625 &= \log 10 - 4 \log 2 \\ &= 1 \\ &- 1{\cdot}20412 \\ & \overline{\bar{1}{\cdot}79588} \end{aligned}$$

Science Examination 1868.

Ex. 3. If log 2 = ·30103, to find, without the tables, the logarithm of ·0000025.

$$\begin{aligned} \log 4 = 2 \log 2 &= {\cdot}60206 \\ {\cdot}0000025 &= \tfrac{1}{4} \times {\cdot}00001 \\ \log {\cdot}0000025 &= \log {\cdot}00001 - \log 4 \\ &= -5 - {\cdot}60206, \end{aligned}$$

or if the decimal part be positive in its usual manner, the logarithm required is

$$\bar{6}{\cdot}39794.$$

Ex. 4. If $x = 2{\cdot}0946$, to find the value of $x^3 - 2x - 4$.

$$\begin{aligned} \log 2{\cdot}0946 &= {\cdot}321\,1011 \\ \log (2{\cdot}0946)^3 &= {\cdot}963\,3033 \\ \log 9{\cdot}1897 &= {\cdot}963\,3013 \\ & \qquad\quad 20 \\ \text{prop. part for } 4 & \qquad\quad 19 \end{aligned}$$

$$\therefore\ x^3 = 9{\cdot}18974,$$
$$\text{and } 2x+4 = 8{\cdot}1892.$$
$$\therefore\ x^3-2x-4 = 1{\cdot}00054.$$

Ex. 5. Calculate to four places of decimals $\dfrac{\frac{5}{8} \text{ of } {\cdot}31416}{\sqrt{{\cdot}93}}$.

If the expression be represented by u,

$$8u = \frac{1{\cdot}5708}{\sqrt{{\cdot}93}}.$$

$$\begin{aligned}
\log 1{\cdot}5708 &= {\cdot}196\,1209\ (13)\\
\log {\cdot}93 &= \bar{1}{\cdot}968\,4829\\
\log \sqrt{{\cdot}93} &= \bar{1}{\cdot}984\,2415\\
\text{ar. comp. } \log \sqrt{{\cdot}93} &= 10{\cdot}015\,7585\\
\log (8u) &= {\cdot}211\,8794\\
\log 1{\cdot}6288 &= {\cdot}211\,8678\\
& \qquad\qquad 116\\
\text{prop. part for } 4 & \qquad\qquad 107
\end{aligned}$$

$\therefore\ 8u = 1{\cdot}62884$, $u = {\cdot}2036$ to four places of decimals.

Science Examination 1868.

Ex. 6. To find the number of digits in $\dfrac{3^{20} \times 5^{15}}{2^{11}}$.

The expression presented is equivalent to $\dfrac{3^{20} \times 5^{26}}{10^{11}}$.

$$\therefore \text{ its } \log = 20 \log 3 + 26 \log 5 - 11,$$
$$= 15 + \text{decimal}.$$

$\therefore$ the expression has 16 integral digits.

Ex. 7. To compute $\left(\dfrac{1}{1{\cdot}04}\right)^{12}$.

$$\log \left(\frac{1}{1{\cdot}04}\right)^{12} = -12 \log 1{\cdot}04 = \bar{1}{\cdot}7956004,$$

$$\text{whence } \left(\frac{1}{1{\cdot}04}\right)^{12} = {\cdot}62458.$$

Ex. 8. To find by logarithms the number of digits in 3^{15}, and the number of ciphers between the decimal point and the first significant digit of the decimal which is equal to $\frac{1}{3^{15}}$.

This question will be resolved by ascertaining what are the characteristics of the logarithms of 3^{15} and $\frac{1}{3^{15}}$.

Now from the tables $\qquad \log 3 = \cdot 4771213$

$$\therefore \log 3^{15} = 15 \log 3 = 7 \cdot 156 \ldots .$$

$$\therefore 3^{15} \text{ consists of 8 digits.}$$

Again,
$$\log \frac{1}{3^{15}} = -15 \log 3$$
$$= -7 \cdot 156 \ldots .$$
$$= \bar{8} \cdot 843 \ldots .$$

$\therefore$ there are 7 ciphers between the decimal point and the first significant digit of the decimal representing $\frac{1}{3^{15}}$.

Ex. 9. Find a third proportional to ·00063 and 8·795.

Science Examination, 1864.

The third proportional required is $\frac{(8 \cdot 795)^2}{\cdot 00063}$. (Algebra 308)

$$\begin{array}{rrl}
\text{Now} \quad \log 8\cdot795 = & \cdot944 & 2358 \\
\log (8\cdot795)^2 = & 1\cdot888 & 4716 \\
\log \cdot00063 = & \bar{4}\cdot799 & 3405 \\
\text{its ar. compt.} = & 13\cdot200 & 6595 \\
\therefore \log \text{ of third proportional} = & 5\cdot089 & 1311 \\
\log 122780 = & 5\cdot089 & 1276 \\
 & & 35 \\
\text{prop. part } 1 & & 35
\end{array}$$

$$\therefore \text{third proportional} = 122781.$$

Use of Logarithms in Computing Compound Interest.

49. Logarithms are brought into use in computing compound interest of money, and in questions respecting annuities, fines, and reversions, where compound interest is taken into account.

Where interest is supposed to be paid yearly, the meaning of compound interest is that the amount at the end of any year becomes the principal for the year ensuing. If 1*l.* be the principal at the beginning of any year, this amounts at the end of the year to 1*l.*, together with the interest on 1*l.* at the rate per cent. supposed. Let R denote this sum unto which a principal of 1*l.* grows at the end of one year. Compound interest therefore has the effect of multiplying the principal at the beginning of any year by this quantity R, to make the principal at the beginning of the next year.

Hence if 1*l.* be the principal placed at interest, it amounts at the end of 1, 2, 3, . . . years to R, R^2, R^3, . . . pounds.

If £P be the principal placed at interest, it amounts at the end of 1, 2, 3, . . . years to PR, PR^2, PR^3, . . . or in general terms, the amount of £P at the end of n years is PR^n.

In calculating R^n logarithms come into use.

50. Ex. To find the amount of 1*l.* at 4 per cent. compound interest, when it has been at interest 24 years.

Here R being 1*l.*, with the yearly interest thereon is 1·04.

The required amount then is $(1 \cdot 04)^{24}$.

$$\log 1 \cdot 04 = \cdot 0170333$$
$$\log (1 \cdot 04)^{24} = \cdot 4087992$$
$$\log 2 \cdot 5633 = \cdot 4087994$$

∴ the amount is 2·5633*l.*

Obs.—This result, obtained by use of logarithms of seven places of decimals only, is not to be trusted beyond the third decimal place.

51. *Examples for Practice.*

1.	The amount of	1*l.*	in 13	years at	5	per cent.	is	1·8857*l.*
2.	"	"	" " 15	"	3	"	"	1·558*l.*
3.	"	"	" " 16	"	$4\frac{1}{2}$	"	"	2·022237*l.*
4.	"	"	" " 37	"	$4\frac{1}{3}$	"	"	4·805*l.*
5.	"	"	125*l.* " 10	"	$3\frac{1}{2}$	"	"	176*l.* 6*s.* 6*d.*

to the nearest penny.

52. If M denote the amount of a principal P at the end of n years, the equation $M = PR^n$ enables us to determine the sum of money which has to be laid out at compound interest, that in a specified number of years it may reach a given amount.

Ex. What sum of money at 6 per cent. compound interest will amount to 1000*l.* in 12 years?

$$P = \frac{M}{R^n} = \frac{1000}{(1·06)^{12}}.$$

$$\begin{aligned} \log 1000 &= 3·000\,0000 \\ \log 1·06 &= \;\;·025\,3059 \\ \log (1·06)^{12} &= \;\;·303\,6708 \\ \log P &= 2·696\,3292 \\ \log 496·97 &= 2·696\,3302 \end{aligned}$$

$\therefore$ the sum required is 496·97*l.* or 496*l.* 19*s.* 5*d.*

53. The number of years can also be found after which a sum placed at compound interest will exceed a given amount.

$$\text{Since } MR^n = M,$$

$$n \log R + \log P = \log M,$$

$$\therefore n = \frac{\log P - \log M}{\log R}.$$

Ex. After how many years will 100*l.* exceed 1000*l.* at 6 *per cent.* compound interest?

$$n = \frac{\log 1000 - \log 100}{\log 1 \cdot 06}$$

$$= \frac{1}{\cdot 0253059}$$

$$= 39 \cdot \ldots$$

Hence at the end of the 39th year the amount will be short of 1000*l.*; at the end of the 40th year the amount will exceed 1000*l.*

54. *Examples for Practice.*

1. At $4\frac{1}{2}$ per cent. compound interest a sum of money will first amount to more than double itself after 16 years.

2. After how many years, at 5 per cent. compound interest, will 100*l.* amount to more than 600*l.*? *Ans.* After 37 years.

55. If interest is payable half yearly, or at other divisions of a year, R must be taken as 1*l.* with the interest which it bears in such interval of a half-year, or other assigned term, and n must be, not the number of years, but the number of payments of interest which fall due.

Reversions.

56. The value of a reversion, or of a sum of money to be paid after a specified time, is the sum which at compound interest will amount in that time to the sum of which the reversion is secured.

If a reversion of 1000*l.* after 10 years have elapsed is purchased, the price to be paid now is the sum which in 10 years will amount to 1000*l.* Supposing the rate of interest to be 4 per cent., this sum is $\frac{1000}{(1 \cdot 04)^{10}} = \frac{1000}{1 \cdot 48024}$,

$$= 675l.\ 10s.$$

Annuities.

57. An annuity is a defined sum of money payable year by year on the same given day. The money required to purchase an annuity is the sum of the present values of the several expected payments to which the claim is thus obtained.

Let A be the annuity or sum to be paid at the end of each year, i.e. at each recurrence of the day of purchase. Let R, as before, be 1*l.* with the year's interest upon it.

The sum A is expected at the end of a year. The right to this is now worth $\frac{A}{R}$, because the sum $\frac{A}{R}$ placed at interest will amount in a year to A.

Another sum A is expected at the end of two years. The right to this is now worth $\frac{A}{R^2}$, because $\frac{A}{R^2}$ will amount to A in two years.

Another sum A is expected at the end of three years. The right to this is now worth $\frac{A}{R^3}$.

Thus the present value of each payment is estimated; and the present value of the annuity, or money required to purchase it, if it is to last n years, or be paid n times, is

$$\frac{A}{R}+\frac{A}{R^2}+\dots+\frac{A}{R^n}.$$

Let P represent the present worth of such an annuity, or the sum which paid now will purchase the annuity, so that

$$P=\frac{A}{R}+\frac{A}{R^2}+\dots+\frac{A}{R^{n-1}}+\frac{A}{R^n}.$$

$$\therefore\ PR=A+\frac{A}{R}+\frac{A}{R^2}+\dots+\frac{A}{R^{n-1}}.$$

$$\therefore\ PR-P=A-\frac{A}{R^n}=A\left(1-\frac{1}{R^n}\right).$$

$$\therefore\ P=\frac{A}{R-1}\left(1-\frac{1}{R^n}\right).$$

58. Ex. 1. If the annuity is $300l.$ yearly, to continue 20 years, interest being taken to be 3 per cent. per annum.

The purchase money is $\frac{300}{\cdot 03}\left\{1-\left(\frac{1}{1\cdot 03}\right)^{12}\right\}l.$ which will be found by use of logarithms to be $4463\cdot 23l.$

Ex. 2. Find the sum which will purchase 15 annual payments of $20l.$ each, the first payment to be made at the end of a year after the purchase, and money being considered to bear $3\frac{1}{2}$ per cent. compound interest. *Ans.* $230l.$ $6s.$ $11\frac{1}{2}d.$

59. The purchase money of a perpetual annuity or freehold is the sum which, at the supposed rate of interest of money, gives a yearly interest equal to the annuity or expected rent of the freehold. The purchase money, divided by the annual income, gives what is called the number of years' purchase for which the property is obtained.

Ex. If an annuity, or other source of perpetual guaranteed income, is $60l.$ yearly, and money is supposed to command 4 per cent. interest, the property would be purchased by that sum of money whose annual interest is $60l.$, namely, $1,500l.$ This would be called 25 years' purchase, since $1,500l.$ is 25 times the yearly income.

Fines for Leases.

60. If property is leased at any rent below its annual value, the fine to be paid for the lease is the present value of all such abatements of rent.

If property is leased for n years at $Al.$ yearly below its value, the fine to be paid for the lease is the value of these n several abatements of rent of $Al.$ each, and is therefore

$$\frac{A}{R-1}\left(1-\frac{1}{R^n}\right)(57).$$

Ex. 1. An estate worth $200l.$ a year is leased for 10 years at a yearly rent of $100l.$ a year, money bearing 4 per cent. interest. The abatement A is here $100l.$, and the sum to be paid for the lease is $811l.$

Ex. 2. If an estate worth 100*l.* yearly is leased for 20 years without any rent being received, the fine to be paid for the lease is, at 3 per cent. interest, 1487*l.* 15*s.*

The same principle applies to determine the fine for renewal of an existing lease, since this is no more than purchasing a fresh lease for a prescribed period.

61. The following question on the increase of population is very similar to a question of compound interest.

One person out of 46 is said to die every year in England, and one out of 33 to be born. If there were no emigration, in how many years would the population double itself?

Consider 46×33 or 1518 persons living at the beginning of any year. Of these, by the statement of the question, 33 die and 46 are born, so that 1518 at the beginning of the year, in the course of the year advances to 1531. Thus population in every year increases by the multiplier $\frac{1531}{1518}$. In n years then it is increased by the multiplier $\left(\frac{1531}{1518}\right)^n$. Wherefore if in n years the population is doubled,

$$\left(\frac{1531}{1518}\right)^n = 2, \quad n = \frac{\log 2}{\log 1531 - \log 1518} = 81{\cdot}3.$$

Hence in 81 years the population will not be doubled, in 82 years it will be more than doubled.

62. The exposition which has here been given of logarithms has been adopted in order to make them intelligible to readers who have not viewed exponents in their full generality, and want to use logarithms in their practical applications. Hereafter it may be needful to consider logarithms from a different point of view, in which, if $a^x = n$ be an equation, x is defined to be the logarithm of n to the base a.

TRIGONOMETRY.

CHAPTER I.

1. An angle, the inclination of two straight lines to one another which meet but are not in the same direction, has its size expressed by a measurement derived from the right angle. Since all right angles are equal in size, the right angle thus furnishes a standard of reference whereby other angles may be compared. The ninetieth part of a right angle is called a degree. A degree is subdivided into sixty equal parts called minutes. A minute is subdivided into sixty equal parts called seconds. Then the size of an angle is expressed by the number of degrees, minutes, and seconds which it contains. The signs $^\circ$, $'$, $''$, are used to denote, for brevity, degrees, minutes, and seconds respectively. Thus $29^\circ\ 18'\ 53''$ means an angle containing 29 degrees, 18 minutes, and 53 seconds.

2. Angles thus represented may be added, subtracted, multiplied, or divided, like any other concrete quantities in Arithmetic.

If to	$23^\circ\ 17'\ 53''$
there be added	$14^\circ\ 56'\ 28''$
the result is	$38^\circ\ 14'\ 21''$.

If from	$23^\circ\ 17'\ 53''$
there be taken	$14^\circ\ 56'\ 28''$
there remains	$8^\circ\ 21'\ 25''$.

$7^\circ\ 15'\ 32''$ multiplied by 13 gives $94^\circ\ 21'\ 56''$,
$94^\circ\ 21'\ 56''$ divided by 13 gives $7^\circ\ 15'\ 32''$.

3. In France and other parts of the Continent it is usual to measure angles by a decimal division. A right angle being made, as with us, the starting-point, the hundredth part of it is called a grade, and this unit of angular measure is divided into tenths, hundredths, &c. The superior convenience is obvious when the addition, subtraction, multiplication, or division of angles is effected by operations in decimals.

The representative of an angle in one measure is easily converted into its representative in the other.

Ex. 1. If an angle is $23^\circ\ 18'$, what is its representative in French measure to hundredths of a grade?

$$\begin{array}{lll}
\text{Since } 90^\circ & \text{correspond to} & 100 \text{ grades} \\
\quad 1^\circ & \quad ,, & \frac{10}{9} \quad ,, \\
\qquad 1' & \quad ,, & \frac{1}{54} \quad ,, \\
\therefore\ 23^\circ & \quad ,, & 25\cdot555 \quad ,, \\
\qquad 18' & \quad ,, & \cdot333 \quad ,, \\
\therefore\ 23^\circ\ 18' & \quad ,, & 25\cdot89 \text{ grades}
\end{array}$$

to the second place of decimals.

Ex. 2. If an angle contain 56·84 grades, how is it expressed in degrees, minutes, and seconds?

$$\begin{array}{ll}
1 \text{ grade being} & \cdot9 \text{ degrees} \\
\therefore\ 56\cdot84 \text{ grades is} & 51\cdot156 \quad ,, \\
& \underline{\quad 60} \\
& 9\cdot36 \text{ minutes} \\
& \underline{\quad 60} \\
& 21\cdot6 \text{ seconds.}
\end{array}$$

$\therefore$ the angle is represented to the nearest second by

$$51^\circ\ 9'\ 22''.$$

4. Though angles, as will be seen hereafter, admit of unlimited magnitude by the opening of the straight lines which form them, in the present elementary course angles will never be supposed to exceed two right angles or 180°.

From 0° up to this size angles will be supposed to admit every degree of magnitude.

5. *Def.*—The complement of an angle is the result when the angle is subtracted from 90°. In other words, an angle and its complement added together make 90°, or a right angle.

Thus 23° 18′ 56″ is the complement of 66° 41′ 4″, and 66° 41′ 4″ is the complement of 23° 18′ 56.″ If an angle exceeds 90° its complement is negative. Thus 109° 27′ has −19° 27′ for its complement.

The angle 45°, or half a right angle, is equal to its complement.

The angle 30° has 60° for its complement, and 60° has 30° for its complement.

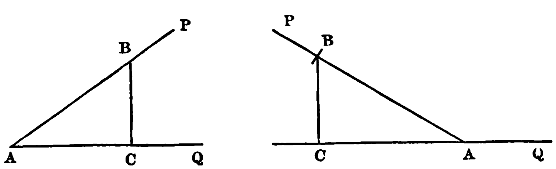
6. *Def.*—The supplement of an angle is the result when the angle is subtracted from 180°. In other words, an angle and its supplement together make 180°, or two right angles.

Thus 23° 18′ 58″ is the supplement of 156° 41′ 2″, and 156° 41′ 2″ is the supplement of 23° 18′ 58″.

The angle 90°, a right angle, is equal to its supplement.

7. In French measure the complement and supplement of an angle are found by subtracting the representative of the angle from 100 or 200 grades respectively.

Thus 34·27 grades is the complement of 65·73 grades,
134·27 „ „ supplement „



8. *Def.*—If PAQ be any angle formed by the straight lines AP, AQ meeting in A, in either of these lines take any point B, and from B draw BC perpendicular to the other line

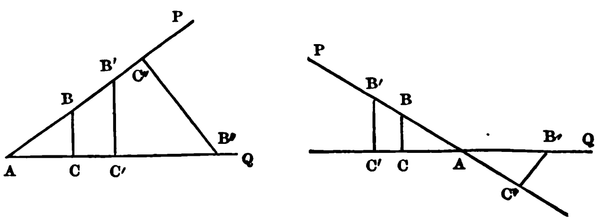
produced if necessary. The ratio of the length of BC to the length of AB is the sine of the angle PAQ.

Since a ratio can be expressed by a quotient or fraction, it may therefore be stated that the sine of the angle PAQ is $\frac{BC}{AB}$.

If when AB, for instance, is taken 10 inches in length, BC is found to be 3·46 inches, the sine of PAQ is ·346.

Since the greater angle of a triangle has the greater side opposite to it, and the angle BAC is less than the right angle BCA, BC is less than AB, and the sine is always a proper fraction, or less than unity.

9. The point B in the definition is stated to be 'any point,' yet the value of the sine of PAQ is the same,



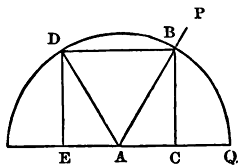
whatever point in either of the containing lines be made the point from which the perpendicular is dropped upon the other. For if any other point B' had been taken in AP, and the perpendicular $B'C'$ dropped on AQ or QA produced, or if the point B'' were taken in AQ and the perpendicular dropped on AP or PA produced, all the triangles thus formed, BAC, $B'AC'$, $B''AC''$, having the angle at A the same in all, and having another angle a right angle, are equiangular. Therefore they are all similar (*Euclid*, vi. 4), and the sides opposite to the equal angles in them are proportionals.

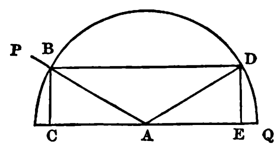
$$\therefore \frac{BC}{AB} = \frac{B'C'}{AB'} = \frac{B''C''}{AB''}.$$

Now any one of these ratios is the sine of the angle PAQ. Therefore an angle has one and only one sine.

10. For brevity an angle is often denoted by the single letter which marks the point where the containing lines meet. Thus the angle PAQ would be called the angle A. The abbreviation 'sin' is also used for 'the sine of angle.' The sine of PAQ would therefore be written sin A.

11. While an angle has but one sine, the same proper fraction has more than one angle of which it is the sine.





If PAQ be an angle, and $\frac{BC}{AB}$ be its sine formed by the definition of (8), with centre A and radius AB describe a circle. Through B draw BD parallel to AQ, meeting the circle again in D. From D draw DE perpendicular to AQ or QA produced.

$$\begin{aligned} &\text{Now since } AB = AD, \\ \therefore\ &\text{the angle } ABD = \text{the angle } ADB. \\ &\text{But since } BD \text{ is parallel to } AQ, \\ &\left.\begin{aligned} &\text{the angle } ABD = \text{the angle } BAC \\ \text{and}\ &\quad ,,\quad ADB = \quad ,,\quad DAE \end{aligned}\right\}, \\ \therefore\ &\text{the angle } BAC = \quad ,,\quad DAE. \end{aligned}$$

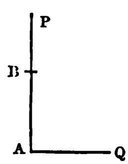
Hence from the angles at E and C being right angles, the triangles BAC, DAE, have two angles in the one equal to two angles in the other, each to each, and a side AB in one equal to a side AD in the other, therefore the other sides of the triangle are equal, each to each, or $BC=DE$.

Hence $\frac{BC}{AB}$, the sine of PAQ, is equal to $\frac{DE}{AD}$, the sine of DAQ; or the two angles PAQ, DAQ, have the same sine.

These two angles, which have the same sine, are supplementary each to the other. For since the angle BAC is equal to the angle DAE, PAQ and DAQ are together equal to two right angles.

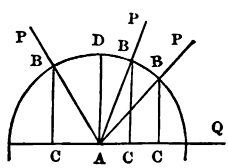
12. When angles are considered to extend beyond two right angles, it will be found that the same sine belongs to a wider range of angles than the two here considered. As far as Trigonometry is now treated, it will suffice for the reader to know this fact, that an angle and its supplement have the same sine.

13. Hence when an angle is given, it will have a certain determinate proper fraction which is its sine, and when any proper fraction is presented there will be two angles, supplementary each to the other, of which it is the sine.



14. If the angle PAQ is a right angle, BA is itself perpendicular to AQ and the ratio $\frac{BC}{AB}$, which is by definition the sine, becomes equal to unity, or $\sin 90^\circ = 1$.

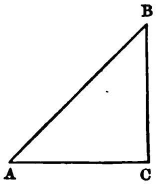
If the angle PAQ shrinks down to zero, BC becomes zero while AB may remain finite. Hence the ratio $\frac{BC}{AB} = 0$, or $\sin 0^\circ = 0$, and $\sin 180^\circ = 0$. (12)



15. If PAQ be any angle, with centre A and any radius AB, describe a semicircle having AQ in its bounding diameter, and by dropping the perpendicular BC on this bounding diameter the sine of the angle PAQ will be $\frac{BC}{AB}$. Suppose AB to move round gradually, and thus to form various angles from AQ. Since in all of them AB is the same in length, the magnitude of the sine is in this view made to depend on the length of BC alone. Hence as the angle increases towards a right angle, the sine gradually increases and $= 1$ when the angle becomes

a right angle. As the angle increases beyond a right angle, BC decreases and the sine decreases down to zero.

16. If ABC be a triangle wherein ACB is a right angle, the two remaining angles BAC, ABC, are together equal to a right angle (*Euclid*, i. 32), and each is therefore the complement of the other.



Now $\frac{BC}{AB}$ is the sine of the angle BAC,

$\frac{AC}{AB}$ " " ABC.

$$\text{Also } BC^2 + AC^2 = BA^2. \quad (\textit{Euclid}, \text{ i. } 47.)$$

$$\therefore \quad \left(\frac{BC}{AB}\right)^2 + \left(\frac{AC}{AB}\right)^2 = 1.$$

Hence the square of the sine of an angle, and the square of the sine of its complement, together make 1.

17. *Def.*—The sine of the complement of an angle is called the cosine of the angle, and is written in the abbreviated form 'cos.' Hence this last result may be written

$$\text{Sin}^2 A + \cos^2 A = 1.$$

18. *Obs.*—Sin ^2A or (sin A)^2 expresses the result of squaring the sine of A. It is not to be confused with the sine of the angle A^2.

19. Let the triangle ABC be one wherein ACB is a right angle and BAC is 45°, or half a right angle. Hence ABC, its complement, is 45° also, or the angle BAC and its complement are equal and their sines are the same.

$$\text{Hence } \sin^2 BAC + \sin^2 BAC = 1 \ (16),$$

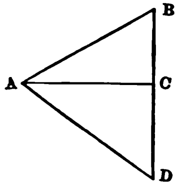
$$\sin BAC = \frac{1}{\sqrt{2}} = \tfrac{1}{2}\sqrt{2}$$

$$= \cdot 707 \text{ to three places of decimals.}$$

In the case, therefore, where BAC is 45°,

$$\sin BAC \text{ or } \sin 45° = \frac{1}{\sqrt{2}} = \cdot 707.$$

20. In the case where BAC is 30°, and consequently ABC, its complement, is 60° (5), produce BC to D, making CD equal to CB, and join AD.



Then, since the two sides BC, CA, are equal to the two sides DC, CA, each to each, and they contain equal angles, viz. right angles, therefore the angle DAC is equal to BAC. Therefore BAD is double of BAC or is 60°. Also AB is equal to AD, whereby the angle ABD is equal to the angle ADB, and each of them is therefore 60°. The triangle being equiangular is consequently equilateral, and AB being equal to BD is double of BC.

$$\therefore \sin BAC = \frac{BC}{AB} = \tfrac{1}{2},$$

$$\text{or } \sin 30° = \tfrac{1}{2} = \cdot 5.$$

21. Since 60° is the complement of 30° (5),

$$\sin^2 60° = 1 - \sin^2 30 \ (16),$$

$$= 1 - \tfrac{1}{4} = \tfrac{3}{4},$$

$$\therefore \sin 60 = \frac{\sqrt{3}}{2} = \cdot 866 \text{ to three places of decimals.}$$

22. The sine of an angle and that of its supplement being the same, it follows from the preceding results that

$$\sin 135° = \sin 45° = \frac{1}{\sqrt{2}} \ (19) = \cdot 707,$$

$$\sin 150° = \sin 30° = \tfrac{1}{2} = \cdot 5,$$

$$\sin 120° = \sin 60° = \frac{\sqrt{3}}{2} = \cdot 866.$$

23. *Caution.*—These instances will suffice to guard the reader from supposing that the sines of angles are proportional to the angles themselves. Thus while the sine of 30° is $\frac{1}{2}$, the sine of its double 60° is $\frac{\sqrt{3}}{2}$, the sine of its triple 90° is unity, the sine of 120° is $\frac{\sqrt{3}}{2}$.

24. *Def.*—Let the same construction be made which was adopted in defining a sine (8). The ratio of the length of BC to AC is the tangent of the angle PAQ, AC being affected with the algebraical sign $+$ or $-$, as the point C lies in AQ or QA produced respectively.

This use of the negative sign accords with the explanation of its meaning in (*Algebra*, 7), whereby if a distance taken in one direction from a point is accounted positive, a distance taken in the contrary direction from the same point may be accounted negative.

The point C will lie in AQ if PAQ is less than a right angle, but in QA produced if PAQ exceeds a right angle. Hence the tangent of an angle less than a right angle is positive, but the tangent of an angle between one and two right angles is negative.

Since there is no restriction on the relative magnitude of BC and AC, the tangents of angles admit all degrees of magnitude. As was explained in the case of the sine (9), the tangent of the angle PAQ has the same value whatever point B in one of the containing lines be adopted for the purpose of defining it, since though various right-angled triangles may be formed they will all be similar, and the ratio of corresponding sides in them will be the same. Thus, an angle has one and only one tangent. For brevity the term 'tan' is used to mean the tangent, and the tangent of PAQ is written tan A.

25. An angle and its supplement have tangents of the same magnitude but contrary in sign.

If the construction in (11) be made, since BC is equal to DE and AC to AE, BC has to AC the same ratio as to length that DE has to AE. But since AC and AE are opposite in sign, the ratios $\frac{BC}{AC}$ and $\frac{DE}{AE}$ are of the same magnitude but contrary in sign. These are the tangents of the angles PAQ and DAQ. These angles, it has been seen, are supplementary one to the other.

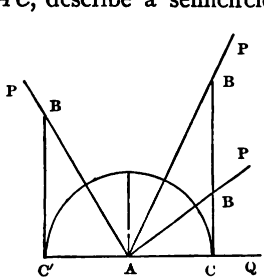
Regarding then at present angles as not extended beyond two right angles, we see that an angle and its supplement have tangents the same in magnitude but opposite in sign.

26. Hence when an angle is given it will have a certain determinate number which is its tangent, and when any number affected with either algebraic sign is presented there will be a certain determinate angle less than two right angles, of which it is the tangent.

27. If the angle PAQ is a right angle, AC disappears, and the tangent of PAQ would be the ratio of a finite line AB to one of no magnitude. The tangent, therefore, of a right angle is beyond numerical representation (*Algebra*, 29).

If the angle PAQ shrinks down to zero, BC becomes zero along with it while AC may remain finite, and the ratio $\frac{BC}{AB}$ is 0, or tan 0° = 0. Also tan 180° = 0.

28. If PAQ be any angle, with centre A and any radius AC, describe a semicircle having AQ in its bounding diameter. At C and C', the ends of this diameter, draw tangents to the circle, perpendicular to AC or AC', and let AP meet one of these tangents in B. Then the tangent of the angle PAQ is $\frac{BC}{AC}$.



Let AP move round gradually and form various angles from AQ. Since in all of them AC has the same length, the magnitude of the tangent depends on BC alone, its algebraic sign on the direction AC or AC'. Hence as the angle increases towards a right angle, the tangent gradually increases and is estimated as positive. When [illegible]omes a right angle, BC is beyond measurement in

comparison with AC, and the tangent of a right angle cannot be numerically represented.

As the angle increases beyond a right angle, BC decreases, and the tangent decreases in magnitude, while it is negative in sign by reason of AC' being accounted negative after AC is made positive.

29. If as in (19) the angle BAC is 45°, ABC is 45° likewise, and $AC = BC$, and the tangent of 45° being $\frac{BC}{AC}$ is unity. Hence also $\tan 135° = -1$. (25)

30. In the case of (20), where the angle BAC is 30°, BA, we have seen, is double of BC.

$$\therefore\ AC^2 = AB^2 - BC^2 = 4BC^2 - BC^2 = 3BC^2,$$

$$\therefore\ \tan 30° = \frac{BC}{AC} = \frac{1}{\sqrt{3}} = \tfrac{1}{3}\sqrt{3} = \cdot5773503,$$

to seven places of decimals.

Also BAC being 30°, ABC is 60°, and $\frac{AC}{BC}$ being the tangent of ABC, $\tan 60° = \sqrt{3} = 1\cdot7320508$, to seven places of decimals.

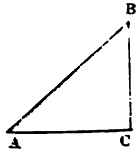
31. The tangent of an angle and its supplement being the same with contrary signs,

$$\tan 150° = -\tfrac{1}{3}\sqrt{3} = -\cdot5773503,$$
$$\tan 120° = -\sqrt{3} = -1\cdot7320508.$$

32. As was observed of the sine, the tangent does not increase in proportion to the angle.

33. The tangent of the complement of an angle is called its cotangent, and is written 'cot.' Thus cot A means the tangent of the angle which is the complement of A.

34. The tangent of an angle and the tangent of its complement are reciprocals each of the other.



If ABC be a triangle wherein C is a right angle, BAC and ABC are complementary, and $\tan BAC = \frac{BC}{AC}$,

$$\tan ABC = \frac{AC}{BC}.$$

Hence $\tan A = \frac{1}{\cot A}$, or $\tan A \cot A = 1$.

Ex. $\sqrt{\tan^3 37^\circ \tan 53^\circ} = \sqrt{\tan^2 37^\circ}$
$= \tan 37^\circ$.

35. Though these definitions of the cosine and the cotangent are given, no calculations will be introduced in this book which require them to be used. There are other ratios also among the sides of the right-angled triangle used in (8), called the secant and the cosecant of the angle A, but no attention will be called to them now, because the object of this book is to conduct the reader in the most easy and direct way to the power of performing calculations by trigonometry.

These quantities, $\sin A$, $\cos A$, $\tan A$, &c., are called goniometrical or trigonometrical functions of the angle A. The word goniometrical is derived from two Greek words which mean 'the measure or size of an angle.'

CHAPTER II.

TABLES OF GONIOMETRICAL FUNCTIONS.

36. The sine and the tangent are the two principal of those several ratios between the sides of the right-angled triangle ABC, which are called 'Goniometrical or Trigonometrical functions' of the angle CAB. Tables are constructed by methods which cannot be explained now, wherein the sines and tangents of all angles, at intervals of minutes, are recorded from 0 to 90°. Hereby the sine and tangent of any angle less than a right angle, given in degrees and minutes, is written down from the tables at once. The sine or tangent of any angle between one and two right angles is known from the recorded sine or tangent of its supplement, which is an angle less than a right angle.

Thus, for example, the tables give

$$\sin 23^\circ\ 18' = \cdot 3955455,$$
$$\tan 54^\circ\ 18' = 1\cdot 3916473,$$

whence also

$$\sin 156^\circ\ 42' = \cdot 3955455,$$
$$\tan 125^\circ\ 42' = -1\cdot 3916473.$$

Generally the sines and tangents of angles are interminable decimals, as it has been seen that the sin of 60° and the tangent of 30° are. The tables usually give these decimals to the seventh place. To save space in the tables, where the sine or tangent is wholly a decimal, the decimal point is not printed.

37. When the table of sines and tangents is constructed at intervals of minutes, and an angle less than 90° is presented which does not contain an exact number of minutes, but is expressed with seconds besides degrees and minutes,

then its sine or tangent is to be found by the method of proportional parts, just as that method is used with logarithms (*Log.* 17). If the table of sines or tangents be examined, it will be observed that throughout the greater part of it the difference between successive records is nearly the same through several lines, or the function is growing in proportion to the angle. It is assumed therefore that throughout the interval of a minute, the growth of the function is likewise at a uniform rate. Suppose now sin 23° 18′ 35″ is required. We have in the tables,

$$\sin 23^\circ\ 19' = \cdot 3958127$$
$$\sin 23^\circ\ 18' = \cdot 3955455.$$

Between these, which differ by ·0002672, the required sine lies.

At this part of the tables then when the angle is increased by a minute, or 60″, the sine is increased by ·0002672. We have to find the proportionate increase of the sine when the angle is increased by 35″. This increase will be

$$\frac{35}{60} \times \cdot 0002672 = \cdot 0001559.$$

$$\therefore\ \sin 23^\circ\ 18' \text{ being } \cdot 3955455,$$
$$\text{and the addition for } 35'' \quad ,, \quad \cdot 0001559,$$
$$\sin 23^\circ\ 18'\ 35'' = \cdot 3957014.$$

This is likewise the sine of 156° 41′ 25″ (11).

38. The same method is applicable to find the tangent of an angle which has seconds besides degrees and minutes.

Thus, if tan 52° 13′ 29″ be required,

$$\text{since } \tan 52^\circ\ 13' = 1 \cdot 2899669,$$

and the difference for 60″ is ·0007752,

$$\therefore \text{ difference for } 29'' = \cdot 0003747.$$
$$\therefore\ \tan 52^\circ\ 13'\ 29'' = 1 \cdot 29003416.$$
$$\text{Hence } \tan 127^\circ\ 46'\ 31'' = -\ 1 \cdot 2903416.$$

Exceptions to the application of proportional parts.

39. It is the foundation of the method of proportional parts that the differences in the tables between several successive

records are nearly the same. When this is not the case, the rule of proportional parts cannot be safely applied. Now, if the table of tangents be examined, it will be seen that when the angle approaches a right angle, the differences of the tangents not only become large, but vary rapidly from line to line. Under these circumstances the rule of proportional parts cannot be safely applied, the tangents of angles containing seconds cannot be found with certainty from the tables, and recourse must be had to methods which cannot at present be given. Where a method of computation brings in such an angle, and requires that its tangent is to be found from the tables, it will be necessary to take some other process wherein this difficulty shall not appear.

Again, whenever the angle is such that the sine changes with extreme slowness for a change of the angle, the sine of an angle containing seconds cannot be safely found, when a limited number of decimals is used. This is the case when the angle is nearly a right angle. If the table be examined it will be seen that from

$$\begin{aligned} \sin 89^\circ\ 42' &= \cdot 9999863, \\ \text{to } \sin 89^\circ\ 43' &= \cdot 9999878, \\ \text{the rise is only } &\cdot 0000015. \end{aligned}$$

Hence $\cdot 0000001$, the smallest recognised quantity when we work to seven places, corresponds to a change of $4''$ in the angle, and the sines of angles differing by less than $4''$ cannot be distinguished one from another. Thus when the angle is close upon a right angle, angles differing by two or three seconds will have to seven places of decimals no difference in their sines.

40. If a number less than unity, to seven places of decimals, is given as a sine, or any number with a decimal to seven places is given as a tangent, the corresponding angle can generally be found. If the number given is a record in the tables, the angle is known at once. If, as is more commonly the case, the number given is between two successive

records in the table, its value is found by the method of proportional parts.

Ex. 1. Let ·3007932 be a sine proposed.

The corresponding angle required is between $17^\circ\ 30'$ and $17^\circ\ 31'$.

$$\begin{aligned} \text{Since the given number is}\quad & \cdot3007932 \\ \text{and } \sin 17^\circ\ 30' = {} & \underline{\cdot3007058} \\ \text{there is an excess over the last record}\quad & \cdot0000874. \end{aligned}$$

Now the angle advances a minute, or $60''$, by an increase of ·0002774 in the sine.

$$\begin{aligned} \therefore \text{ required advance} &= \frac{\cdot0000874}{\cdot0002774} \times 60 \text{ seconds} \\ &= 19 \text{ seconds.} \end{aligned}$$

$\therefore$ the angle required is $17^\circ\ 30'\ 19''$.

This is one of the two angles belonging to the given sine. The angle $162^\circ\ 29'\ 41''$ belongs to it as well.

The method may be expressed in the form of a rule. Subtract from the given number the one next below it in the tables. Multiply the result by 60 and divide by the difference in the tables. The quotient is the additional number of seconds to the angle in the tables next below that required.

Ex. 2. To find the angle whose tangent is ·8347266,

$$\begin{array}{rr} & \cdot8347266 \\ \tan 39^\circ\ 51' = & \underline{\cdot8346481} \\ & \cdot0000785 \\ & \underline{60} \\ \cdot0004937) & \cdot000471\ \ (9 \\ & \underline{44433} \\ & 2667 \end{array}$$

$\therefore$ $39^\circ\ 51'\ 10''$ is the angle required.

41. From the remarks of (39) and (40) it will be seen that when an angle is nearly a right angle it cannot be safely found from its sine or tangent; not from its tangent because the principle of proportional parts cannot be then trusted;

not from its sine, because the sine is then changing so slowly that when taken only to seven places of decimals it cannot express minute differences in the angle.

42. *Examples for Practice.*

1. sin 28° 13′ 15″ = ·4728712.
2. sin 156° 14′ 24″ = ·4029064.
3. tan 12° 16′ 5″ = ·2124856.
4. tan 132° 17′ 51″ = −1·0990821.

Each of these examples may be reversed, and be made an exercise in finding the value of an angle from its given sine or tangent.

43. *Tables of the Logarithms of Goniometrical Functions.*

Since, in computations into which goniometrical functions enter logarithms are very often brought into use, tables are therefore constructed which give the logarithms of the functions at intervals of minutes. But in these tables there is this alteration made in the logarithms: every one of them is increased by 10 before it is printed. The motive for this is twofold.

(1.) Since all sines are less than 1, and the tangents likewise of angles up to 45° (8, 29), the logarithms of those sines and tangents have negative characteristics. By adding 10 the characteristics are all made positive, and the tables look more compact in print than they would look if so many negative characteristics appeared in them.

(2.) There is a more important object in this, that if several logarithms have to be combined by addition or subtraction, some with positive and some with negative characteristics, there is more time taken, and more liability to make a mistake than there would be when all the characteristics are positive.

This excess of the tabular above the true logarithm of a function has to be kept in mind, and allowed for at a proper stage of the computation.

44. The tabular logarithm of a trigonometrical function is distinguished by the capital letter L, while the true logarithm is written with the small initial letter.

$$\text{Thus} \quad \text{L} \sin 20^\circ\ 15' = 10 + \log \sin 20^\circ\ 15'.$$

The table of sines gives

$$\sin 20^\circ\ 15' = \cdot 3461171.$$

If the logarithm of this number be taken, it is found to be

$$\bar{1} \cdot 539223.$$

The table accordingly gives

$$\text{L} \sin 20^\circ\ 15' = 9 \cdot 539223.$$

45. When an angle is presented in degrees and minutes, the tabular logarithm of its sine or tangent can be written down at once from the tables. If it has any additional seconds the addition for them is calculated by the method of proportional parts, unless it be at a part of the tables where the logarithms vary by differences which change rapidly. It will be observed that this is the case for the logarithmic tangents of angles which are near to a right angle, and for the logarithmic sines and tangents of very small angles. Hence it is inconvenient to have recourse to methods of calculation which introduce the logarithmic functions of any such angles.

46. Again, if a number is presented as the tabular logarithmic sine or tangent of an angle, and the very number appears in the tables, the angle can at once be assigned, with the reservation in the case of the sine, that an angle and its supplement are equally admissible. If the number lies between two successive records in the tables, the additional seconds of the angle are found by the method of proportional parts, unless it be at a part of the table where the principle on which proportional parts rest is not satisfied, viz. that the logarithm is increasing by differences which are nearly the same for some successive records, or if it be at a part of the table where the logarithm is changing

so slowly that it fails to discriminate seconds. These cases of exception are when the angle is very small or very nearly a right angle.

47. Ex. 1. To find the L sin 37° 18′ 34″.

$$\begin{array}{rrl}
& \text{L sin } 37^\circ\ 18' = & 9{\cdot}782\,4643 \\
\text{Difference for } 60'' = & {\cdot}0001658 & \\
& 34 & \\
\hline
& 6632 & \\
& 4974 & \\
\hline
60)\,{\cdot}0056372 = & & {\cdot}000\,0940 \\
\therefore\ \text{L sin } 37^\circ\ 18'\ 34'' = & & 9{\cdot}782\,5583
\end{array}$$

48. Ex. 2. To find the angle whose L sin is 9·583 4276.

In this table the record immediately below this is

$$\begin{array}{ll}
\text{L sin } 22^\circ\ 31' = & 9{\cdot}583\,1445 \\
\text{difference} & {\cdot}000\,2831 \\
& \quad 60 \\
\hline
{\cdot}0003046)\,{\cdot}0016986\ (55 & \\
\quad 15230 & \\
\hline
\quad 17560 & \\
\quad 15230 & \\
\hline
\quad 2330 &
\end{array}$$

∴ the required angle to the nearest second is

22° 31′ 56″ or 157° 28′ 4″ (11).

49. To exemplify the manner in which the true logarithms of the sines and tangents are altered into the tabular logarithms, and the alteration afterwards recognised, suppose it is requisite in some computation to find the numerical value of

$$x = \sqrt{\tan 44^\circ\ 11'\ 53'' \times (\sin 63^\circ\ 18')^3}.$$

$$\text{First } \log x = \tfrac{1}{2}\{\log \tan 44^\circ\ 11'\ 53'' + 3 \log \sin 63^\circ\ 18'\}.$$

$$= \tfrac{1}{2}\{\text{L tan } 44^\circ\ 11'\ 53'' - 10 + 3\ \text{L sin } 63^\circ\ 18' - 30\}.$$

$$\begin{aligned}
\text{Now L}\sin 63^\circ\ 18' &= 9{\cdot}951\,0320 \\
& \phantom{= 29{\cdot}951\,032}3 \\
& \overline{29{\cdot}853\,0960} \\
\text{L}\tan 44^\circ\ 11'\ 53 &= 9{\cdot}987\,8400 \\
& \overline{39{\cdot}840\,9360} \\
& -\ 40 \\
& \overline{\bar{1}{\cdot}840\,9360} \\
\therefore \log x &= \bar{1}{\cdot}920\,4680 \\
\therefore x &= {\cdot}83266.
\end{aligned}$$

50. The following example is to show the method of dealing with the tangents of obtuse angles.

To find the value of $\tan^3 127^\circ\ 18' \times \sqrt[3]{\sin 134^\circ\ 15'}$.

The required value being denoted by x,

$$\begin{aligned}
x &= \tan^3 127^\circ\ 18'\ .\ \sqrt{\sin 134^\circ\ 15'}, \\
-x &= \tan^3 52^\circ\ 42' \times \sqrt{\sin 45^\circ\ 45'}, \\
\log(-x) &= 3\log\tan 52^\circ\ 42' + \tfrac{1}{2}\log\sin 45^\circ\ 45' \\
&= 3\ \text{L}\tan 52^\circ\ 42' - 30 + \tfrac{1}{2}\ \text{L}\sin 45^\circ\ 45' - 5.
\end{aligned}$$

$$\begin{aligned}
\text{L}\tan 52^\circ\ 42' &= 10{\cdot}118\,1614 \\
& \phantom{= 10{\cdot}118\,161}3 \\
& \overline{30{\cdot}354\,4842} \\
\text{L}\sin 45^\circ\ 43' &= 9{\cdot}855\,0961 \\
\tfrac{1}{2}\ \text{L}\sin 45^\circ\ 45' & \ \ 4{\cdot}927\,5481 \\
3\ \text{L}\tan 52^\circ\ 42' + \tfrac{1}{2}\ \text{L}\sin 45^\circ\ 45' &= 35{\cdot}282\,0323 \\
& \ \ 35 \\
\log(-x) &= {\cdot}282\,0323 \\
\therefore -x &= 1{\cdot}914398 \\
x &= -1{\cdot}914398.
\end{aligned}$$

51. *Examples.*

1. The following examples may be used for practice, either to find the L sin or L tan of the angles presented, or to find the angles on supposition of the L sin or L tan being given.

$$\begin{aligned}
&\text{L sin } 23^\circ\ 23'\ 23'' = 9{\cdot}5987722.\\
&\text{L sin } 64^\circ\ 24'\ 18'' = 9{\cdot}9551449.\\
&\text{L tan } 62^\circ\ 18'\ 11'' = 10{\cdot}2798872.\\
&\text{L tan } 55^\circ\ 36'\ 9'' = 10{\cdot}1645321.\\
&\text{L sin } 57^\circ\ 30'\ 6'' = 9{\cdot}9260372.\\
&\text{L sin } 126^\circ\ 37'\ 30'' = 9{\cdot}9044761.\\
&\text{L tan } 84^\circ\ 37'\ 23'' = 11{\cdot}0263094.\\
&\text{L tan } 36^\circ\ 6'\ 17'' = 9{\cdot}8629290.
\end{aligned}$$

The angle whose L sin is 9·9938045 is 80° 20′ 42″ or 99° 39′ 18″.

" " " 9·7706640 is 36° 8′ 20″ or 143° 51′ 40″.

" " L tan is 10·0629627 is 49° 8′ 19″

" " " 9·8694731 is 36° 31′.

2. The angle whose tangent is 3·21976 to the nearest minute is 72° 45′.

3. The angle whose sine is $\frac{1}{4}$ is to the nearest second 14° 28′ 39″.

4. The angle whose sine is $\frac{1}{3}$ to the nearest second is 19° 28′ 16″.

5. Find by the tables to the nearest second the angle whose sine is $\frac{1}{6}$. *Ans.* 9° 35′ 39″, or 170° 24′ 21″.

6. Find to the nearest second the angle whose tangent is $\frac{31}{25}$. *Ans.* 51° 6′ 56″

7. Find by the tables the angle whose sine is $\sqrt{\frac{7}{15}}$.
Ans. 43° 5′ 19″, or 136° 54′ 41″.

8. Sin 36° × tan 54° = ·654.

9. If the angle A is 72° 16′ 52″, $\sqrt[3]{(\sin A)} = {\cdot}98393$.

10. Compute to the fourth place of decimals the value of

$$\frac{1}{\sin^2 79^\circ\ 36'(1+\sin 10^\circ\ 24')}.$$ *Ans.* ·8756.

11. Compute the value of $\sqrt{\frac{16{\cdot}6667}{\tan 15^\circ}}$. *Ans.* 7·8868.

12. If $A = 129^\circ\ 18'$, $(\tan A)^{\frac{1}{5}} = -1{\cdot}0409$.

CHAPTER III.

SOLUTION OF RIGHT-ANGLED TRIANGLES.

52. The student being now qualified to find from the tables the sine and tangent of any angle, with a few exceptions, between 0 and 180°, as well as the logarithms of these sines and tangents, also to determine an angle from its given sine, tangent, log sine or log tangent, he is now to see the utility of these trigonometrical functions in computing the size of the remaining parts of a right-angled triangle from certain of its parts whose size is given. This process is called the solving, or the solution of the triangle.

53. In a right-angled triangle, the right angle being a fixed determinate angle, there remain two angles and the three sides, which are open to variation in magnitude under these restrictions, viz. that

(1) The two angles together make a right angle,

(2) Of the three sides, any two are together greater than the third. (*Euclid*, i. 20, 32).

Now, if there be given either

(1) Two sides of the triangle, or

(2) One side and one of the acute angles,

the triangle with these given parts can be constructed on paper by use of a scale and protractor, and one triangle only can be formed.

It will now be our object not to draw on paper, but to compute in numbers, the size of those remaining parts of a right-angled triangle from two given parts.

We may either have given

(1) two sides of the triangle,

and then there are to be found the third side and the angles, or

(2) one side and one of the acute angles,
and then there are to be found the two remaining sides and the remaining acute angle.

54. The principle of the calculations which are now to be explained will be readily grasped, if it is observed that a table of sines or tangents is in fact a tabulation of one side of a right-angled triangle in terms of another. If C be the right angle in the triangle ABC, the tangent of BAC being $\frac{BC}{AC}$ is an exhibition of the length of BC in parts of AC as a unit; the sine of BAC, or $\frac{BC}{AB}$, is an exhibition of the length of BC in parts of AB as a unit.

55. I. The two sides given may be either

(1) the two shorter sides,

(2) a shorter side and the longest side, or hypotenuse as it is called.

Ex. 1. The angle ACB being the right angle, let $AC = 5$ feet, $BC = 4$ feet.

$$\begin{aligned} \therefore AB &= \sqrt{AC^2 + BC^2} \\ &= \sqrt{41} = 6{\cdot}403 \text{ feet}, \\ \tan A &= \frac{BC}{AC} = \frac{4}{5} = {\cdot}8. \\ \therefore A &= 38^\circ\ 39'\ 59''\ (40), \\ B &= 51^\circ\ 20'\ 1'', \end{aligned}$$

and the parts of the triangle are completely computed.

In this example the numbers are so small that logarithms have not been called into use. In this next example the larger numbers will make logarithms an assistance.

56. Ex. 2. Let $AC = 3297$ feet, $BC = 5463$ feet.

$$\begin{aligned} AB &= \sqrt{(3297)^2 + (5463)^2} \\ &= \sqrt{40914578} = 6380{\cdot}8 \text{ feet}, \\ \tan A &= \frac{5463}{3297}. \end{aligned}$$

$$
\begin{aligned}
\text{Log tan } A &= \log 5463 - \log 3297 \\
\text{L tan } A &= 10 + \log 5463 - \log 3297 \\
&= 10
\end{aligned}
$$

$$
\begin{array}{lr}
\log 5463 & 3{\cdot}737\,4312 \\
\hline
 & 13{\cdot}737\,4312 \\
\log 3297 & 3{\cdot}518\,1189 \\
\hline
 & 10{\cdot}219\,3123 \\
\text{L tan } 58^\circ\,53' = & 10{\cdot}219\,2253 \\
\hline
 & {\cdot}000\,087 \\
 & 60 \\
\hline
{\cdot}0002856) & {\cdot}000\,0522\ (18 \\
 & 2856 \\
\hline
 & 23640 \\
 & 22848 \\
\hline
 & 792
\end{array}
$$

$$
\begin{aligned}
\therefore A &= 58^\circ\ 53'\ 18''. \\
\text{Hence } ABC &= 31^\circ\ \ 6'\ 42''.
\end{aligned}
$$

If the angle A had been first found, it would have been possible to deduce from it the side AB in manner following:

$$
\text{Sin } A = \frac{BC}{AB}.
$$

$$
\therefore AB = \frac{BC}{\sin A}.
$$

$$
\begin{aligned}
\therefore \log AB &= \log BC - \log \sin A \\
&= \log BC + 10 - \text{Log} \sin A
\end{aligned}
$$

$$
\begin{array}{lr}
10 + \log BC = & 13{\cdot}737\,4312 \\
\hline
\text{L sin } 58^\circ\ 53' = & 9{\cdot}932\,5330 \\
\text{addition for } 18'' = & {\cdot}000\,0229 \\
\hline
 & 9{\cdot}932\,5559 \\
\hline
\therefore \log AB = & 3{\cdot}804\,8753 \\
\log 6380{\cdot}8 & 3{\cdot}804\,8751 \\
\therefore AB = & 6380{\cdot}8 \text{ feet.}
\end{array}
$$

This latter method of finding AB may look less laborious, but in computation it is always preferable to deduce results, where it is possible, from the given elements directly, instead of making one result a stepping-stone for obtaining another. The reason of this preference is obvious, because in the latter method a mistake made in the first result is propagated into the second. The values of all the logarithms and trigonometrical functions which we employ are not exact but approximate (36), and this want of exactness may enlarge the error when one result is obtained through a preceding one, and not independently from the original given parts.

57. Ex. 3. The shorter sides of a right-angled triangle are 10, 12, respectively. Find the third side or hypotenuse, and the sines of the two acute angles.

Science Examination 1867.

Let ABC be the triangle wherein C is the right angle, $AC = 10$, $CB = 12$, the two given sides.

$$\begin{aligned} \therefore\ AB \text{ the hypotenuse} &= \sqrt{AC^2 + BC^2} \\ &= \sqrt{100 + 144} \\ &= \sqrt{244} \\ &= 15{\cdot}62 \text{ feet.} \end{aligned}$$

$$\sin BAC = \frac{12}{15{\cdot}62},$$

$$\sin ABC = \frac{10}{15{\cdot}62}.$$

Ex. 4. When the hypotenuse and one of the shorter sides are given, let

$$AB \text{ the hypotenuse} = 20 \text{ feet}, \quad AC = 14 \text{ feet.}$$

$$\begin{aligned} \therefore\ BC = \sqrt{AB^2 - AC^2} &= \sqrt{34 \times 6} \\ &= \sqrt{204} = 14{\cdot}29, \end{aligned}$$

to two places of decimals.

$$\sin ABC = \frac{AC}{AB} = \cdot 7.$$

Now $\sin 44^\circ\ 25' = \underline{\cdot 6998711}$

$$\begin{array}{r} \cdot 0001289 \\ 60 \\ \hline \end{array}$$

$$\begin{array}{r} \cdot 0002078)\cdot 007734\ (37 \\ \underline{6234} \\ 15000 \\ \underline{14546} \\ 454 \end{array}$$

$\therefore\ ABC = 44^\circ\ 25'\ 37''.$

$\therefore\ BAC = 45^\circ\ 34'\ 23''.$

58. Ex. 5. In the following example the numbers ar larger, and logarithms are an assistance.

Given AB the hypotenuse $= 8437\cdot 62$ yards,

$AC = 5627\cdot 06$ yards.

$$BC = \sqrt{AB^2 - AC^2} = \sqrt{(AB+AC)(AB-AC)}$$
$$= \sqrt{14064\cdot 68 \times 2810\cdot 56}.$$

log 14064	$= 4\cdot 148$	1089
prop. part for 6		0185
" " 8		0025
log 2810·5	$= 3\cdot 448$	7836
prop. part for 6		92
	$7\cdot 596$	9227
log BC	$= 3\cdot 798$	4614
log 6287·2	$= 3\cdot 798$	4573
		41
prop. part for 6		41

$\therefore\ BC = 6287\cdot 26$ yards.

$$\sin ABC = \frac{AC}{AB} = \frac{5627\cdot 06}{8437\cdot 62}.$$

$L \sin ABC = 10 + \log 5627\cdot 06 - \log 8437\cdot 62.$

$$
\begin{array}{rr}
10+\log 5627 = & 13{\cdot}750\,2769 \\
\text{prop. part for } 6 & 47 \\
\hline
 & 3{\cdot}750\,2816 \\
\hline
\log 8437{\cdot}6 = & 3{\cdot}926\,2189 \\
\text{prop. part for } 2 & 10 \\
\hline
 & 3{\cdot}926\,2199 \\
\hline
\therefore \text{L} \sin ABC = & 9{\cdot}824\,0617 \\
\text{Now L} \sin 41^\circ\ 49' = & 9{\cdot}823\,9626 \\
\hline
 & {\cdot}000\,0991 \\
 & 60 \\
\hline
\end{array}
$$

$$
\begin{array}{r}
{\cdot}0001419){\cdot}005946\ (41 \\
5676 \\
\hline
2700 \\
1419 \\
\hline
1281
\end{array}
$$

$$\therefore\ ABC = 41^\circ\ 49'\ 42''.$$
$$\therefore\ CAB = 48^\circ\ 10'\ 18''.$$

59. II. When an angle and a shorter side are given, this side may be either the side adjacent to the given angle or the side opposite to it. But in reality these two apparently different cases are but one. For if the angle A be given and the adjacent side AC, since the angle B, the complement of A is known at once, we are started with what are practically the same given parts, whether it be that AC and the angle A, or AC and the angle B, are given.

Ex. 1. Let $AC = 5$ feet, and the angle $A = 38^\circ\ 40'$.

$$\frac{BC}{AC} = \tan A.$$
$$\begin{aligned} BC &= AC \tan 38^\circ\ 40' \\ &= 5 \times {\cdot}80011963 \\ &= 4 \text{ feet.} \end{aligned}$$

The angle $ABC = 51^\circ\ 20'$.

$$AC = AB \,.\, \sin ABC,$$

$$AB = \frac{AC}{\sin ABC} = \frac{5}{\cdot 780794}$$

$$= 6{\cdot}7 \text{ feet},$$

whereby all the parts of the triangle are determined.

60. Ex. 2. In a triangle ACB where ACB is a righ angle let $AC = 89646$ yards; and the angle BAC b 35° 25′ 25″. To find BC.

$$BC = 89646 \times \tan 35° \; 25' \; 25''.$$

$$\begin{array}{rl}
\text{L} \tan 35° \; 25' = & 9{\cdot}851\,9312 \\
\text{addition for } 25'' = & \quad\quad\;\; 1115 \\
\log 89646 = & \underline{4{\cdot}952\,5309} \\
\log BC = & 4{\cdot}804\,5736 \\
\log 63763 = & \underline{4{\cdot}804\,5687} \\
 & \quad\quad\quad\;\; 49 \\
\text{prop. part for } 7 & \quad\quad\quad\;\; 48
\end{array}$$

$$\therefore \; BC = 63763{\cdot}7 \text{ yards}.$$

61. Ex. 3. Given the hypotenuse $AB = 29745$ feet, and the angle $A = 31° \; 4' \; 18''$.

$$\text{The angle } ABC = 58° \; 55' \; 42''.$$

$$\frac{BC}{AB} = \sin A.$$

$$\therefore \; BC = AB \sin A.$$

$$\log BC = \log AB + \log \sin A$$

$$= \log AB + \text{L} \sin A - 10.$$

$$\begin{array}{rl}
\text{Now } \log AB = & 4{\cdot}473\,4140 \\
\text{L} \sin 31° \; 4' = & 9{\cdot}712\,6792 \\
\text{prop. part for } 18'' & \quad\quad\;\; \underline{0627} \\
\log BC = & 4{\cdot}186\,1559 \\
\text{Now } \log 15351 = & \underline{4{\cdot}186\,1367} \\
 & \quad\quad\quad\; 192 \\
\text{prop. part for } 6 & \quad\quad\quad\; \underline{169} \\
 & \quad\quad\quad\;\; 23 \\
\text{,,} \quad \text{,,} \quad 8 & \quad\quad\quad\;\; 23
\end{array}$$

$$\therefore \; BC = 15351{\cdot}68 \text{ feet}.$$

Again $\frac{AC}{AB} = \sin ABC$.

$\therefore \log AC = \log AB + \text{L} \sin ABC - 10$.

$$
\begin{array}{lrr}
\log AB = & 4\cdot473 & 4140 \\
\text{L} \sin 58^\circ\ 55' = & 9\cdot932 & 6854 \\
\text{prop. part for } 42'' & & 0533 \\
\hline
\log AC = & 4\cdot406 & 1527 \\
\text{Now } \log 25477 = & 4\cdot406 & 1483 \\
\hline
 & & 44 \\
\text{prop. part for } 2 & & 34 \\
\hline
 & & 100 \\
\text{"} \quad \text{"} \quad 6 & & 103
\end{array}
$$

$\therefore AC = 25477\cdot26$ feet.

62. *Examples for Practice.*

In these examples no results will be exact (36), but true to the decimal place or to the subdivision of the angle to which they are presented.

I. (1) When the two shorter sides are given.

1. These sides being 3584·63 and 3301·75 feet in length, the opposite angles of the triangle are 42° 38′ 51″ and 47° 21′ 8″, the hypotenuse is 4873·5 feet.

2. The sides being 328 and 388 yards long, the opposite angles of the triangle are 40° 12′ 36″ and 49° 47′ 24″, and the hypotenuse is 508 yards.

(2) When the hypotenuse and another side are given.

3. The hypotenuse being 278·469 yards, and another side 217·496 yards, the angles of the triangle are 51° 21′ 22″, 38° 38′ 38″, and the third side 173·898 yards.

II. (1) When an acute angle and one of the shorter sides are given.

4. If the given angle be 41° 42′ 43″ and the adjacent side 2934 feet, the other sides of the triangle are 2615·74 and 3931·33 feet.

5. If the given angle be $48° 41' 53''$ and the side opposite to it 3456·89 yards, the other sides of the triangle are 3037·46 and 4601·77 yards.

(2) When an acute angle, and the hypotenuse are given.

6. The given angle being $35° 29' 52''$ and the hypotenuse 165 feet, the lengths of the other sides are 96 and 134 feet.

7. If an acute angle be given, $37° 18'$ and the hypotenuse 573·8 feet, the shorter sides of the triangle are 347 and 456·5 feet.

The reader will have it in his power to make from these examples more exercises for himself, if this be his desire, by taking in any one of the triangles just presented two parts as the data, and computing from these the other parts of the triangle.

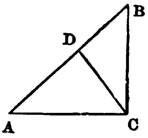
63. Although the calculations of Trigonometry are the means whereby a triangle is solved with exactness, it is nevertheless a good practice to lay down the triangle on paper by drawing instruments, and the required parts can then roughly be found by measuring them with the scale or protractor. This is a convenient practice with a view to check the trigonometrical determinations, because if the computer should find his results seriously at variance with his measures of the figure, he may be warned to review his work, and detect the mistake which has caused the disagreement.

64. *Area of a Right-angled Triangle.*

The area of a right-angled triangle, being half the area of a rectangle of the same base and altitude (*Euclid*, i. 41), is half the product of the sides which are at right angles to one another. If these two sides are given the area is known at once. If other two parts are given, either one of the acute angle and the hypotenuse, or one of the acute angles and

a side, in either case the perpendicular sides become known, and the area can be found.

65. To find the perpendicular distance of the right angle C in the triangle ABC from the opposite side AB.



Let CD be this distance.

If the sides AC and CB are given.

Twice the area of the triangle is the rectangle AC, CB, and also the rectangle AB, CD.

$$\therefore\ CD \cdot AB = AC \cdot CB,$$

$$CD = \frac{AC \cdot CB}{AB}$$

$$= \frac{AC \cdot CB}{\sqrt{AC^2 + CB^2}}.$$

If the hypotenuse AB and one other side AC is given,

$$CD = \frac{AC \cdot CB}{AB} = \frac{AC\sqrt{AB^2 - AC^2}}{AB}.$$

If the hypotenuse AB and the angle A be given, the angle B its complement being at once also known,

$$CD = AC \sin A$$

$$= AB \cdot \sin B \cdot \sin A.$$

If a side AC and the angle A be given,

$$CD = AC \sin A.$$

66. *Examples for Practice.*

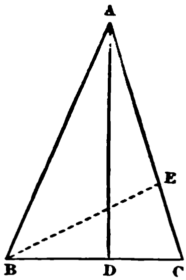
1. If an angle of a right-angled triangle is 28° 13′ 54″ and the side opposite to it is 27 yards long, the area is 679 square yards.

2. If the hypotenuse of a right-angled triangle is 16 feet in length, and one of the angles of the triangle is 36° 18′, the area is 61 square feet.

3. If the hypotenuse of a right-angled triangle is 73 yards and its area is 968 yards, its acute angles are 23° 18′ and 66° 42′.

67. *Solution of an Isosceles Triangle.*

An isosceles triangle is reducible to two right-angled triangles, and can be solved if an angle and a side, or if two unequal sides be given in magnitude.



If ABC be an isosceles triangle where the sides AB, AC are equal to one another, and consequently the angles ABC, ACB are equal to one another. If AD be drawn perpendicular to BC, it will bisect this side BC at right angles.

1. When an angle is given, the other angles are also at once known, since the three angles of a triangle are two right angles.

Then $BC = 2BD = 2AB \sin \frac{A}{2} = 2AB \sin (90° - B)$,

and BC is known from AB, or AB from BC.

2. When two unequal sides are given

$$\sin \frac{A}{2} \text{ or } \sin (90° - B) = \frac{BD}{AB} = \frac{BC}{2AB},$$

and the angles can be found.

68. *Area of an Isosceles Triangle.*

The area of the isosceles triangle is $\frac{1}{2} AD \,.\, BC$

$$= \tfrac{1}{2} \,.\, AB \,.\, BC \sin B,$$

or if a perpendicular BE be drawn upon AC,

$$\text{the area} = \tfrac{1}{2} AC \,.\, BE,$$
$$= \tfrac{1}{2} AC \,.\, AB \sin A,$$
$$= \tfrac{1}{2} AB^2 \sin A.$$

69. *Examples for Practice.*

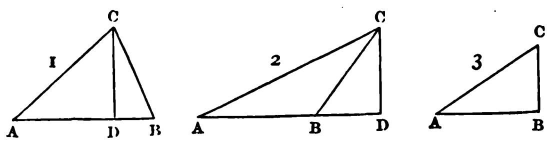
1. If the lengths of the sides of an isosceles triangle are 183, 183, and 197 yards, its angles are 65° 6′ and 57° 27′.

2. The equal sides of an isosceles triangle being each 200 feet, and the equal angles each 30° 15′, the remaining side of the triangle is 345·534 feet in length.

CHAPTER IV.

OBLIQUE-ANGLED TRIANGLES.

70. We proceed now with triangles wherein it is not a given fact that any angle is a right angle. Such angles are for distinction sometimes called oblique-angled triangles. For their treatment the following two propositions are required.



I. Let ABC be a triangle. Since two at least of its angles are acute, let CAB be one of them, and let CBA be either acute as in Figure 1, or obtuse as in Figure 2, or a right angle as in Figure 3. In Figures 1 and 2 draw CD perpendicular to AB, meeting it, produced if necessary, in D.

In Figure 1 or 2,

$$CD = AC \sin A,$$
$$\text{also} = BC \sin B.$$
$$\therefore\ AC \sin A = BC \sin B,$$
$$\text{or}\quad \frac{\sin A}{\sin B} = \frac{BC}{AC}.$$

In Figure 3,

$$BC = AC \sin A,$$
$$\text{and}\ \sin B = 1.$$
$$\therefore\ \frac{\sin A}{\sin B} = \frac{BC}{AC}.$$

Hence, in any triangle the sines of any two angles have the same ratio which the sides opposite to them have, whether

the angles be both acute, or one be obtuse, or one be a right angle. This is expressed by the equations

$$\frac{\sin A}{BC}=\frac{\sin B}{AC}=\frac{\sin C}{AB}.$$

71. II. If ABC be a triangle wherein A and B are acute angles (Fig. 1, 70), and if from the point C a straight line be drawn perpendicular to AB, meeting it in D, and if AC be greater than BC, to prove that

$$\frac{AB}{AC+CB}=\frac{AC-BC}{AD-DB}.$$

Since $BC^2-BD^2=AC^2-AD^2$,
each being equal to CD^2 (*Euclid*, i. 47),

$$\therefore\ AD^2-BD^2=AC^2-BC^2,$$

$$\therefore\ (AD+DB)(AD-DB)=(AC+BC)(AC-BC),$$

$$\text{or } AB\,.\,(AD-DB)=(AC+BC)(AC-BC).$$

$$\therefore\ \frac{AB}{AC+BC}=\frac{AC-BC}{AD-DB}.$$

This proposition is expressed sometimes in words, calling the side AB the base of the triangle, that the base has to the sum of the other sides the same ratio which the difference of these sides has to the difference of the segments of the base.

72. If the angles A and B are not both acute, so that the perpendicular CD falls on AB produced (Fig. 2, 70), if AC be greater than BC it may be similarly proved that

$$\frac{AB}{AC+CB}=\frac{AC-CB}{AD+BD},$$

but this proposition will not at present be called into use.

73. Hence, if the sides of the triangle be given, the lengths of the two segments into which the perpendicular divides the base can be computed.

Let the base be 24 feet and the sides 23 and 25 feet

$$AD - BD = \frac{48 \times 2}{24} = 4,$$

while $AD + BD = 24$.

$\therefore\ 2AD = 28, \qquad AD = 14$ feet,

$\quad\ 2BD = 20, \qquad BD = 10$ feet.

74. *Obs.*—The sides of a triangle ABC usually have their lengths designated by the symbols a, b, c, in the order of the angles to which they are opposite, viz. $BC = a$, $AC = b$, $AB = c$. Then the last propositions may be stated in the forms

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$

$$\frac{c}{a+b} = \frac{a-b}{AD-BD}.$$

75. Methods shall now be given for solving any oblique-angled triangle, by reducing it if need be to two right-angled triangles, and then making its solution result from the solution of these right-angled triangles by the methods already set forth.

76. A triangle has six parts, three angles and three sides. The three angles together make two right angles (*Euclid*, i. 32), and of the sides no one must be so great as the sum of the other two (*Euclid*, i. 20). If, in compliance with these conditions, any values be given at random to any three of the parts, it will be found that, with two exceptions, it is always possible to lay down on paper one, and only one, triangle which has these parts. Thus the assignment of three parts in general makes the triangle determinate.

The three parts may be:

I. Three angles.

II. Two angles and one side, which side may be either:

(1) the side between the two angles, or

(2) a side opposite one of the two angles.

III. An angle and two sides, which sides may be either:

(1) the sides containing the angle between them, or

(2) sides not containing the angle between them.

IV. Three sides.

Each of these four cases and their sub-cases are now to be considered.

77. I. Let three angles be given, together two right angles.

This is one of the two cases where three given parts do not fix and determine the triangle, because an endless number of triangles can be formed all equiangular and similar, any one therefore having the three given parts. In all these triangles the sides have the same ratios among themselves, but there is nothing to fix the actual lengths of any of them. The shape of the triangle is known, but not its magnitude.

78. II. (1) Let two angles be given and the side between them.

Let A and B be the angles given; then the angle $C = 180°-(A+B)$ is at once known. Also, let c denote the given side; then

$$a = c\frac{\sin A}{\sin C},$$

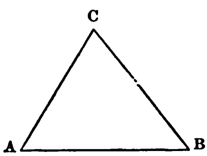
$$b = c\frac{\sin B}{\sin C}.$$

$$\therefore \log a = \log c + \log \sin A - \log \sin C$$

$$= \log c + \text{L} \sin A - \text{L} \sin C \ (43).$$

$$\log b = \log c + \text{L} \sin B - \text{L} \sin C.$$

Thus the sides a and b can be found, and they complete the solution of the triangle.



79. Ex. 1. In the triangle ABC let the angle $CAB = 60°\ 15'$,
„ „ $CBA = 54°\ 30'$,
and the side $AB = 1$ yard.
To find the side AC.

The angle $ACB = 180°-114°\ 45'$

$$= 65°\ 15',$$

$$\frac{AC}{AB} = \frac{\sin 54°\ 30'}{\sin 65°\ 15'},$$

$$AC = \frac{\sin 54°\ 30'}{\sin 65°\ 15'}.$$

$$
\begin{aligned}
\text{L} \sin 54^\circ\ 30' &= 9{\cdot}910\,6860 \\
\text{L} \sin 65^\circ\ 15' &= 9{\cdot}958\,1543 \\
\text{and its arithmetic complement} &= {\cdot}041\,8457 \\
& \quad 9{\cdot}952\,5317 \\
\therefore \log AC &= \bar{1}{\cdot}952\,5317 \text{ (Log 36)} \\
\log {\cdot}89646 & \quad \bar{1}{\cdot}952\,5309 \\
\therefore AC &= {\cdot}89646 \text{ yards.}
\end{aligned}
$$

(2) Let the two angles A and B be given, the third angle $C = 180-(A+B)$ being thereby also known, and let the side a opposite to one of the angles be also given.

$$\text{Then } b = a\frac{\sin B}{\sin A}.$$

$$c = a\frac{\sin C}{\sin A}.$$

$$\log b = \log a + \text{L} \sin B - \text{L} \sin A,$$

$$\log c = \log a + \text{L} \sin C - \text{L} \sin A,$$

and the sides b and c are determined.

80. Ex. 2. Let the angle $A = 67^\circ\ 58'\ 50''$.

„ „ $B = 59^\circ\ 7'\ 44''$,

and the side $BC = 1468$ feet. To find the side AC.

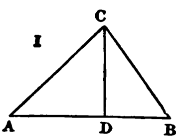
$$
\begin{aligned}
A &= 67^\circ\ 58'\ 50'' \\
B &= 59^\circ\ \ 7'\ 44'' \qquad 180^\circ\ 0'\ 0'' \\
A+B &= \qquad\qquad\quad 127^\circ\ 6'\ 34'' \\
C &= \qquad\qquad\quad 52^\circ\ 53'\ 26''.
\end{aligned}
$$

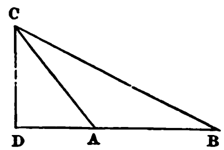
$$
\begin{aligned}
\log a &= 3{\cdot}166\,7261 \\
\text{L} \sin 59^\circ\ 7' &= 9{\cdot}933\,5957 \\
\text{addition for } 44'' &= {\cdot}000\,0554 \\
& \quad 13{\cdot}100\,3772 \\
\text{L} \sin 67^\circ\ 58' &= 9{\cdot}967\,0637 \\
\text{addition for } 50'' &= {\cdot}000\,0426 \\
& \quad 9{\cdot}967\,1063 \\
\therefore \log b &= 3{\cdot}133\,2709 \\
\log 1359{\cdot}16 &= 3{\cdot}133\,2705
\end{aligned}
$$

$\therefore$ b or the side AC is 1359·2 feet,
to the first place of decimals.

81. III. (1) Let the angle A be given and the two sides b, c which contain it.

If the angle A is acute, from C, the end of the shorter of

1 

2 

the containing sides, drop a perpendicular CD on AB the longer of them (Fig. 1).

If the angle A is obtuse, from C, the end of either of the containing sides, drop a perpendicular CD on the other side BA produced (Fig. 2).

Then the angle ACD, being $90°-A$ (Fig. 1),
or $A-90°$ (Fig. 2) is known.

$$\text{Hence } AD = AC \sin ACD,$$
$$\text{and } CD = AC \sin A, \text{ are known.}$$

$$\text{Then } BD = c - AD \text{ (Fig. 1)},$$
$$\text{or } c + AD \text{ (Fig. 2), is known also.}$$

Hence, in the right-angled triangle CDB, there are the two perpendicular sides known in length, from which the remaining parts are to be found by the following formulæ:

$$\tan B = \frac{CD}{DB},$$

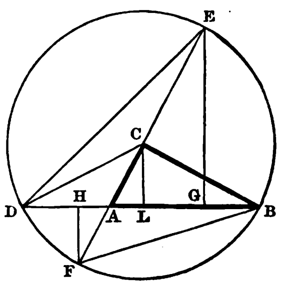
$$CB = \sqrt{CD^2 + DB^2} \text{ or } \frac{CD}{\sin B},$$

$$\text{angle } C = 180° - (A + B).$$

82. The inconvenience of this process lies in its consisting of several steps, so that an accidental error in one of the earlier ones is propagated into those which follow, and also

in its requiring at last the hypotenuse of a right-angled triangle to be found from the two sides, which, when these sides have any large numerical values, is laborious. The following proposition leads to a more ready determination of the two angles, and then of the remaining side.

83. In any triangle the tangent of the semi-difference of any two angles is to the tangent of the semi-sum of these angles as the difference of the sides respectively opposite to them is to the sum of the same sides.



If a, b, c designate the sides of a triangle opposite to the angles A, B, C respectively, this proposition is expressed by the statement:

$$\frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)} = \frac{a-b}{a+b}.$$

With C the angular point of the triangle ABC as centre, and with radius CB the longer of the two sides which contain the angle C, describe a circle $BFDE$. Produce BA to meet the circle in D, and produce AC both ways to meet the circle in F and E. Join DC, BF.

From $\left.\begin{matrix} F \\ E \\ C \end{matrix}\right\}$ draw $\left\{\begin{matrix} FH \\ EG \\ CL \end{matrix}\right.$ perpendicular to BD.

Then ECB being the sum of the angles A and B (*Euc.* i. 32),
EDB is the semi-sum $\frac{1}{2}(A+B)$ (*Euc.* iii. 20);
and DCA being the difference of the angles A and CDA or A and B (i. 32),
DBF is the semi-difference $\frac{1}{2}(A-B)$ (*Euc.* iii. 20).

Now FE being bisected in C,

$$HL = LG,$$
$$\therefore\ BH = DG,$$

$$\text{and } \tan\frac{A-B}{2} = \tan DBF = \frac{FH}{BH},$$

$$\tan\frac{A+B}{2} = \tan EDG = \frac{EG}{DG} = \frac{EG}{BH},$$

$$\therefore\ \frac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}} = \frac{FH}{EG} = \frac{AF}{AE} = \frac{CF-AC}{CE+AC} = \frac{CB-AC}{CB+AC}$$
$$= \frac{a-b}{a+b}.$$

$$\text{Since } \frac{A+B}{2} = 90^\circ - \frac{C}{2},$$

the proposition thus demonstrated may take the form

$$\tan\frac{A-B}{2} = \frac{a-b}{a+b} \cdot \tan\left(90^\circ - \frac{C}{2}\right).$$

[I have not hesitated to offer this proof because I made it myself, although I have lately learned that a proof similar in substance is given by Mr. Todhunter in his 'Trigonometry for Beginners.'—W. N. G.]

84. When this proposition is employed to solve a triangle from the given parts a, b, C, we obtain from this equation the value of $\frac{A-B}{2}$. But $\frac{A+B}{2}$ being known also from its being the complement of $\frac{C}{2}$, the angles A and B are determined.

$$\text{Then } \frac{c}{a} = \frac{\sin C}{\sin A},$$

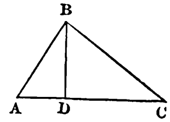
$$\text{or } c = a\frac{\sin C}{\sin A},$$

is an equation which gives the length c.

85. To exemplify and contrast the two methods of solu-

tion, take the case where the given parts are the angle $A = 56° \; 28'$, and the containing sides $b = 327$ feet, $c = 256$ feet.

From B, the end of the shorter of the containing sides, drop a perpendicular BD on AC. The angle $ABD = 33° \; 32'$.



$$
\begin{aligned}
\text{Now } BD &= AB \sin 56° \; 28'. \\
\log AB &= 2{\cdot}4082400 \\
\text{L} \sin 56° \; 28' &= \underline{9{\cdot}9209393} \\
\therefore \log BD &= 2{\cdot}3291793. \\
AD &= AB \sin 33° \; 32'. \\
\log AB &= 2{\cdot}4082400 \\
\text{L} \sin 33° \; 32' &= \underline{9{\cdot}7422710} \\
\log AD &= 2{\cdot}1505110 \\
\log 141{\cdot}42 & \quad 2{\cdot}1505108 \\
\therefore AD &= 141{\cdot}42 \text{ feet,} \\
\therefore CD &= 185{\cdot}58 \;\; \text{,,}
\end{aligned}
$$

$$
\text{Again } \tan C = \frac{BD}{CD}.
$$

$$
\begin{aligned}
\text{L} \tan C &= 12{\cdot}3291793 \\
& \quad \underline{\;\;2{\cdot}2685312} \\
& \quad 10{\cdot}0606481 \\
\text{L} \tan 48° \; 59' &= \underline{10{\cdot}0605818} \\
& \qquad\qquad 663 \\
& \qquad\qquad \underline{\;\;60} \\
& \quad 2551)39980(16 \\
\therefore C &= 48° \; 59' \; 16''.
\end{aligned}
$$

$$
\text{Then } BC = \frac{BD}{\sin C}.
$$

$$
\begin{aligned}
10 + \log BD &= 12{\cdot}3291793 \\
\text{L} \sin \text{C} &= \underline{\;\;9{\cdot}8780821} \\
\log BC &= \;\;2{\cdot}4510972 \\
\log 282{\cdot}55 &= \;\;2{\cdot}4510953 \\
\therefore BC &= 282{\cdot}55 \text{ feet.}
\end{aligned}
$$

If the formula of (83) be employed,

$$\tan\frac{B-C}{2} = \frac{327-256}{327+256}\tan\left(90^\circ - \frac{A}{2}\right)$$

$$= \tfrac{71}{583}\tan 61^\circ\ 46',$$

$$\begin{aligned}
\log 71 &= 1{\cdot}851\,2583 \\
\text{L}\tan 61^\circ\ 46' &= 10{\cdot}270\,0705 \\
\hline
&\ 12{\cdot}121\,3288 \\
\log 583 &= 2{\cdot}765\,6686 \\
\hline
\therefore\ \text{L}\tan\frac{B-C}{2} &= 9{\cdot}355\,6602 \\
\text{Now L}\tan 12^\circ\ 46' &= 9{\cdot}355\,2267 \\
\hline
&\phantom{=1{\cdot}355}\ 4335 \\
&\phantom{=1{\cdot}355\ 43}\ 60 \\
\hline
&\ 5859)260100(44{\cdot}3
\end{aligned}$$

$$\begin{aligned}
\therefore\ \frac{B-C}{2} &= 12^\circ\ 46'\ 44{\cdot}3'', \\
B-C &= 25^\circ\ 33'\ 29'', \\
\text{also } B+C &= 123^\circ\ 32', \\
\therefore\ 2B &= 149^\circ\ \ 5'\ 29'', \\
B &= 74^\circ\ 32'\ 44'', \\
2C &= 97^\circ\ 58'\ 31'', \\
C &= 48^\circ\ 59'\ 15''.
\end{aligned}$$

We can now find BC by either of the following methods:

$$\frac{BC}{AB} = \frac{\sin A}{\sin C},$$

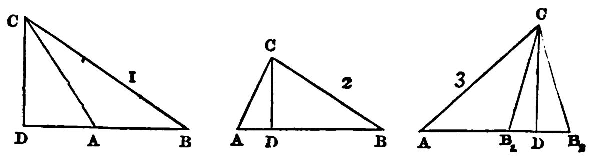
$$\begin{aligned}
\log BA &= 2{\cdot}408\,2400 \\
\text{L}\sin A &= 9{\cdot}920\,9393 \\
\hline
&\ 12{\cdot}329\,1793 \\
\text{L}\sin C &= 9{\cdot}877\,6847 \\
\hline
\log BC &= 2{\cdot}451\,4946, \\
\log 282{\cdot}81 &= 2{\cdot}451\,4948, \\
\therefore\ BC &= 282{\cdot}81 \text{ feet};
\end{aligned}$$

$$\text{or } \frac{BC}{AC} = \frac{\sin A}{\sin B},$$

$$\begin{array}{rl} \log AC = & 2\cdot514\,5478 \\ L \sin A = & \underline{9\cdot920\,9393} \\ & 12\cdot435\,4871 \\ L \sin B = & \underline{9\cdot984\,0061} \\ \log BC = & 2\cdot451\,4810, \\ \log 282\cdot8 & 2\cdot451\,4794 \\ \therefore\ BC = & 282\cdot8 \text{ feet.} \end{array}$$

Different methods of solution will give results with small discrepancies like these, because logarithms are used, not in their exact values, but to a limited number of places of decimals.

86. (2) Let the angle A be given and also the sides a, b, which do not contain it.



From the point C wherein the two given sides meet draw CD perpendicular to AB or AB produced.

$$\text{Now } \sin B = \frac{b}{a} \sin A.$$

Since CDB is a right angle, CBD is less than a right angle,

$$\begin{array}{ll} \therefore & CD \text{ is less than } CB \ (\textit{Euc. } \text{i. } 19.), \\ \text{or} & b \sin A \quad\quad ,, \quad\quad a, \\ \text{and} & \dfrac{b \sin A}{a} \text{ is a proper fraction.} \end{array}$$

There are therefore two angles, supplementary one to the other, which have this fraction for their sine (13), and they

appear as values of the angle CBA of the triangle. The question arises, which is to be adopted, the acute or the obtuse value?

Now, if A is obtuse as in Fig. 1, B must be acute.

Or if A is acute and b less than a, as in Fig. 2, B is less than A (*Euc.* i. 18), and therefore acute.

But if A is acute and b greater than a, as in Fig. 3, the condition of B being greater than A does not settle the difficulty, because either the acute or obtuse value of B satisfies this condition.

Now this is an ambiguity which would have shown itself if we had attempted to lay down a triangle on paper from a given value of the acute angle A and given lengths of the sides AC, CB, where AC is the greater of the two. For when the acute angle BAC is laid down, and AC measured off, then with centre C and radius CB we should proceed to describe a circle whose intersection with AB would fix the point B. But since CB is less than CA this circle intersects AB in *two* points which lie on the *same side* of A. Thus two triangles are formed with the given parts, and the angles B in them are supplemental. For if B_1 and B_2 be the points of intersection, since $CB_1 = CB_2$, the angle CB_1B_2 equals the angle CB_2B_1. But CB_1A_1, CB_1B_2 are two right angles. Therefore CB_1A, CB_2A are two right angles or supplemental one to the other.

In constructing Fig. 1 or 2 the ambiguity does not arise, because the circle from centre C would cut AB or AB produced on different sides of A, and produce only one triangle with the given angle A.

Supposing the acute angle B determined in Fig. 1

$$BD = a \sin DCB = a \sin (90° - B),$$
$$AD = b \sin ACD = b \sin (A - 90),$$

are known, and thence

$$c = BD - AD \text{ is found.}$$

In Fig. 2

$$BD = a \sin(90 - B)$$
$$AD = b \sin(90 - A)$$
and $c = BD + AD$ is found.

In Fig. 3, where two values of B are obtained, when the smaller value is taken,

$$AD = b \sin(90 - A),$$
$$BD = a \sin(90 - B),$$
and $c = BD + AD$ is found.

When the larger or obtuse value is taken,

$$AD = b \sin(90 - A),$$
$$BD = a \sin(B - 90),$$
and $c = AD - BD$ is known.

87. The following is an example of the ambiguous case.

Let the angle $A = 24° \ 17'$,
BC or $a = 300$ feet,
AC or $b = 326$ feet.

$$\begin{array}{rr}
\log b = & 2{\cdot}513\,2176 \\
\text{L} \sin A = & 9{\cdot}614\,1051 \\
\hline
 & 12{\cdot}127\,3227 \\
\log a = & 2{\cdot}477\,1213 \\
\hline
\text{L} \sin B = & 9{\cdot}650\,2014 \\
\text{L} \sin 26° \ 32' = & 9{\cdot}650\,0338 \\
\hline
 & 1676 \\
 & 60 \\
\end{array}$$

$$\begin{array}{r}
2530)100560(39 \\
7590 \\
\hline
24660 \\
22770 \\
\hline
1890
\end{array}$$

∴ the angle B is either

$$26° \ 32' \ 40''$$
$$\text{or } 153° \ 27' \ 20''.$$
$$\therefore C = 129° \ 10' \ 20''$$
$$\text{or } \ 2° \ 15' \ 40''.$$

In the former of these two cases

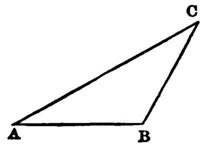
$$\begin{array}{rr}
\text{L} \sin C = & 9{\cdot}889\,3736 \\
& 687 \\
\log a = & 2{\cdot}477\,1213 \\ \hline
& 12{\cdot}366\,5636 \\
\text{L} \sin A = & 9{\cdot}614\,1051 \\ \hline
\log c = & 2{\cdot}752\,4585 \\
\log 565{\cdot}53 = & 2{\cdot}752\,4556 \\
\therefore\ AB \text{ or } c = & 565{\cdot}53 \text{ feet.}
\end{array}$$

In the second case

$$\begin{array}{rr}
\text{L} \sin C = & 8{\cdot}593\,9483 \\
& 2\,1356 \\
\log a = & 2{\cdot}477\,1213 \\ \hline
& 11{\cdot}073\,2052 \\
\text{L} \sin A = & 9{\cdot}614\,1051 \\ \hline
& 1{\cdot}459\,1001 \\
\log 28{\cdot}78 = & 1{\cdot}459\,0908 \\
\therefore\ AB \text{ or } c = & 28{\cdot}79 \text{ feet.}
\end{array}$$

88. In the following example there is a condition introduced which relieves the question from ambiguity.

The base of a triangle is 100 feet. The angle opposite the base is 34° 18′, and the longer of the two sides which contain this angle is 150 feet. What are the other angles of the triangle?



Let AB be the base of the triangle, 100 feet in length, ACB the opposite angle 34° 18′, AC the longer of the containing sides, 150 feet.

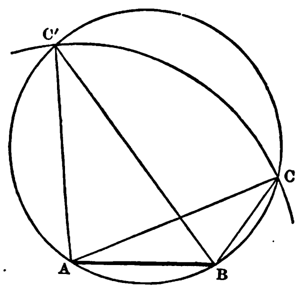
$$\text{Now} \sin B = \frac{AC}{AB} \sin C$$
$$= 1{\cdot}5 \sin 34^\circ\, 18'$$
$$= {\cdot}8452890.$$
$$\therefore\ B = 57^\circ\, 42'\, 10''$$
$$\text{or } 121^\circ\ \ 7'\, 50''.$$

If $B = 57° 42' 10''$, then $A = 87° 59' 50''$.
If $B = 121° 7' 50''$, then $A = 24° 34' 10''$.

Now AC is the longer of the two sides AC, BC. Hence the angle CBA is larger than the angle CAB (*Euc.* i. 18), and the latter solution is to be adopted, inasmuch as the former presents the angle A larger than the angle B. Then the given circumstance of the *longer* side which contains the given angle being 150 feet disposes of the ambiguity, and the other angles of the triangle can be pronounced to be $121° 7' 50''$ and $24° 34' 10''$.

The ambiguity which arises in this problem and the condition which clears it away will be illustrated by constructing the triangle by measurement.

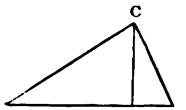
On AB describe a segment of a circle, the angle of the segment being $34° 18'$. Then the vertex of the triangle lies on the circumference of this segment. With centre A, and radius 150 feet on the scale on which AB is 100 feet, describe a circle. This circle, it will be found, cuts the former circumference in two points C and C'. Join AC', AC, BC', BC. Then two triangles ACB, $AC'B$ have been constructed with these common properties :



the base AB is 100 feet,
the vertical angle is $34° 18'$,
a side containing this angle is 150 feet.

In one of them AC, 150 feet, is longer than the other side CB; in the other triangle AC' is shorter than BC'. Since then the question prescribes that 150 feet is the length of the longer of the two sides which contain the given angle, it follows that ACB and not $AC'B$ is the triangle intended.

89. IV. Let the three sides a, b, c be given. Of the three sides let AB be that which is not less than either of the other two, and upon this side drop a perpendicular from the opposite angle meeting AB in D. Suppose that of the two segments AD, AB, the greater is AD,



$$\text{then } AD-DB = \frac{(b-a)(b+a)}{c} \ (71),$$

$$\text{while } AD+DB = c,$$

gives the lengths of AD and DB.

$$\text{Hence } \sin ACD = \frac{AD}{AC},$$

$$\sin BCD = \frac{BD}{BC},$$

determine the angles ACD, BCD, since each of these angles, being an angle of a right-angled triangle, is acute.

$$\text{Then } A = 90°-ACD,$$

$$B = 90°-BCD,$$

become known,

$$\text{and thence } C = 180°-(A+B),$$

which completes the solution of the triangle.

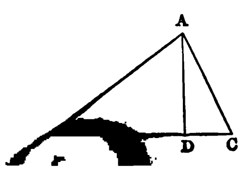
90. Ex. In the triangle ABC let the lengths of the sides be $AB = 1056$, $AC = 924$, $CB = 1716$ yards.

To find the angle C.

Upon BC, the longest side, drop a perpendicular AD from A.

$$\text{Then } BD-CD = \frac{(AB+AC)(AB-AC)}{BC} = \frac{1980}{13},$$

$$\text{while } BD+CD = 1716.$$



$$\therefore\ CD = \frac{10164}{13}$$

$$\sin CAD = \frac{CD}{AC} = \frac{10164}{12012}.$$

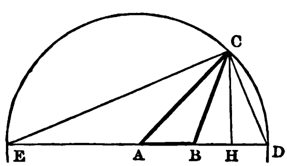
$$\begin{array}{rr|l} \text{L}\sin CAD = & 14{\cdot}007 & 0647 \\ - & 4{\cdot}079 & 6153 \\ \hline & 9{\cdot}927 & 4494 \\ \text{L}\sin 57^\circ\ 47' = & 9{\cdot}927 & 3899 \\ \hline & & 595 \\ & & \ \ 60 \end{array}$$

$$\begin{array}{l} 796)35700(44 \\ \quad\ \ \underline{3184} \\ \quad\ \ \ \ 3860 \\ \quad\ \ \ \ \underline{3184} \\ \quad\ \ \ \ \ \ 676 \end{array}$$

$$\therefore\ CAD = 57^\circ\ 47'\ 45'',$$
$$ACD = 32^\circ\ 12'\ 15''.$$

91. When the numbers are large which express the lengths of the sides of the triangle, the computation of the lengths of the segments of the base is laborious, and the following method is easier. It will depend on an expression for the tangent of half an angle of a triangle in terms of the sides, which shall now be obtained.

In the triangle ABC of the two sides AC, AB which include the angle A, let AC be the greater, and with centre A and distance AC describe a circle cutting AB produced each way in the points D and E. Draw CH perpendicular to ED, and join CD, EC. Then designating the sides of the triangle by a, b, c, we have $EA = AD = AC = b$,



$$BD = b-c,\ EB = b+c.$$

Also the angle $CEA = \frac{1}{2}A$ (*Euc.* iii. 20).

Now since ECD is a right angle,

$$CD^2 = ED \,.\, DH \text{ (\textit{Euc.} vi. 8)}$$
$$EC^2 = ED \,.\, EH.$$

$$\therefore \frac{CD^2}{CE^2} = \frac{DH}{EH} = \frac{b-c-BH}{b+c+BH}$$
$$= \frac{2c(b-c)-2c \,.\, BH}{2c(b+c)+2c \,.\, BH}.$$

Now in the triangle ABC

$$b^2 = c^2+a^2+2c \,.\, BH. \quad (\textit{Euc. ii. 12.})$$
$$\text{or} \quad 2c \,.\, BH = b^2-c^2-a^2,$$
$$\therefore \tan^2\frac{A}{2} = \frac{CD^2}{CE^2} = \frac{2c(b-c)-b^2+c^2+a^2}{2c(b+c)+b^2-c^2-a^2}$$
$$= \frac{a^2-b^2-c^2+2bc}{b^2+c^2+2bc-a^2}$$
$$= \frac{a^2-(b-c)^2}{(b+c)^2-a^2}$$
$$= \frac{(a+b-c)(a+c-b)}{(a+b+c)(b+c-a)}$$
$$\tan\frac{A}{2} = \sqrt{\frac{(a+b-c)(a+c-b)}{(a+b+c)(b+c-a)}},$$

the value of $\tan\frac{A}{2}$ being necessarily positive, since A is less than 180°.

If the angle B is acute, B lies between H and D, and the same result arises, *Euc.* ii. 13 being employed.

92. For practice this formula is most readily employed by introducing the symbol S for the semiperimeter of the triangle, or $2S = a+b+c$.

$$\text{Then } \tan\frac{A}{2} = \sqrt{\frac{(S-b)(S-c)}{S(S-a)}}.$$

93. Since with the figure of (89),

$$AD-BD = \frac{b^2-a^2}{c},$$
$$AD+DB = c,$$
$$\therefore \quad 2AD = \frac{b^2-a^2+c^2}{c}.$$
$$4CD^2 = 4b^2-(2AD)^2$$

$$\begin{aligned}
\therefore\quad 4c^2 \,.\, CD^2 &= 4b^2c^2 - (b^2 - a^2 + c^2)^2 \\
&= (2bc + b^2 - a^2 + c^2)(2bc - b^2 + a^2 - c^2) \\
&= \{(b+c)^2 - a^2\}\,\{a^2 - (b-c)^2\} \\
&= (b+c+a)(b+c-a)(a+b-c)(a-b+c).
\end{aligned}$$

If $a+b+c$, the perimeter of the triangle, be designated by $2S$,

$$4c^2 \,.\, CD^2 = 16S \,.\, (S-a)(S-b)(S-c).$$

$$\text{Now}\quad CD^2 = b^2 \sin^2 A,$$

$$\therefore\; b^2c^2 \sin^2 A = 4\, S(S-a)(S-b)(S-c),$$

$$\sin A = \frac{2}{bc}\sqrt{S(S-a)(S-b)(S-c)}.$$

The positive sign is taken because $\sin A$ is positive.

$$\text{Hence}\ \frac{\sin A}{a} = \frac{2}{abc}\sqrt{S(S-a)(S-b)(S-c)}\,,$$

$$\text{and}\ \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{\sin A}{a} \quad (70)$$

$$= \frac{2}{abc}\sqrt{S(S-a)(S-b)(S-c)}.$$

By these formulæ the angles A, B, C are found by the assistance of logarithms when the sides are given, but they are not so convenient as that of (92) because more separate logarithms are to be taken out of the tables.

94. Let it be given that the sides are

$$\begin{aligned}
a \text{ or } BC &= 583{\cdot}946 \text{ feet} \\
b \text{ or } AC &= 791{\cdot}854 \quad ,, \\
c \text{ or } AB &= \underline{465{\cdot}387} \quad ,, \\
\therefore\; 2S &= 1832{\cdot}187
\end{aligned}$$

$$\begin{aligned}
S &= 916{\cdot}094, & S &= 916{\cdot}094, & S &= 916{\cdot}094 \\
a &= \underline{583{\cdot}946} & b &= \underline{791{\cdot}854} & c &= \underline{465{\cdot}387} \\
\therefore\; S-a &= 322{\cdot}148, & S-b &= 124{\cdot}24\;, & S-c &= 459{\cdot}707.
\end{aligned}$$

$$L \tan \frac{A}{2} = \tfrac{1}{2}\{\log (S-b) + \log (S-c) - \log S - \log(S-a)\} + 10$$

$$\begin{aligned}
\log (S-b) &= 2{\cdot}094\,2614 \\
\log (S-c) &= 2{\cdot}662\,4745 \\
&\phantom{= 2{\cdot}662\,47}66 \\
&\ \overline{4{\cdot}756\,7425} \\
\log S &= 2{\cdot}961\,9381 \\
&\phantom{= 2{\cdot}961\,93}19 \\
\log (S-a) &= 2{\cdot}521\,3212 \\
&\phantom{= 2{\cdot}521\,3}105 \\
&\ \overline{5{\cdot}483\,2717} \\
\log (S-b) + \log (S-c) - \log S - \log(S-a) &= \bar{1}{\cdot}273\,4708 \\
L \tan \frac{A}{2} &= 9{\cdot}636\,7354 \\
L \tan 23^\circ\ 25' &= 9{\cdot}636\,5722 \\
&\phantom{= 9{\cdot}636\,}1632 \\
\text{prop. part for } 27'' &= \phantom{9{\cdot}636\,}1639
\end{aligned}$$

$$\therefore\ \frac{A}{2} = 23^\circ\ 25'\ 27'',\ A = 46^\circ\ 50'\ 54''.$$

So $B = 98^\circ\ 23'\ 26''$,
$C = 34^\circ\ 45'\ 40''$.

95. *Examples for Practice.*

The following examples may produce other exercises by alteration of the data, as was suggested in (62).

I. (1) Given two angles and the side between them.

1. The angles being $38^\circ\ 19'\ 5''$ and $44^\circ\ 18'\ 6''$, and the side between them being 438·25 feet, the other sides are 308·6 and 274 feet.

2. The angles being $56^\circ\ 35'\ 18''$ and $59^\circ\ 42'\ 17''$, and the side [illegible] them being 3279 feet, the other sides are [illegible] feet.

3. Two angles of a triangle being $47° 18' 39''$ and $98° 7'$, and the length of the side between them being 864 feet, the longest side of the triangle is 1507 feet.

4. Two angles of a triangle being $46° 18' 37''$, $59° 12' 41''$, and the side between them being 161 yards, the shortest side of the triangle is 121 yards.

(2) Given two angles and an opposite side.

5. The angles being $42° 18' 17''$, $47° 16' 18''$, and the side opposite to the former 2220·37 feet, the other sides are 3298·77 and 2432·27 feet. This triangle is nearly right-angled.

6. The angles being $59° 15' 18''$, $57° 16' 25''$, and the side opposite to the former 684·937 feet, the other sides are 670·44 and 713 feet.

7. Two angles of a triangle being $86° 17'$, $47° 29' 47''$, and the side opposite the latter being 242 feet, the side opposite the third angle of the triangle is 237 feet.

8. Two angles of a triangle are $29° 22' 34''$, $89° 54' 10''$, and the side opposite to the latter is 501 feet long, find the length of that side of the triangle which is neither the longest nor the shortest of the three. *Ans.* 437 feet.

III. (1) Given an angle and the sides which contain it.

9. If the sides 678·924 and 783·888 feet in length include the angle $62° 18'$, the remaining side is 772·79 feet, and the other angles are $65° 38' 8''$ and $51° 3' 52''$.

10. If the sides 689·536 and 574·927 feet include the angle $53° 53' 53''$, the remaining side is 798 feet and the other angles are $73° 9' 34''$ and $52° 56' 33''$.

11. If in a triangle two sides 23 and 56 yards in length include the angle $143° 29'$, the smallest angle of the triangle is $10° 24' 44''$.

12. Find the smallest angle of the triangle wherein sides 347 and 296 feet in length include the angle $108° 17'$.

Ans. $32° 34' 39''$.

13. Find the third side of a triangle wherein sides 827 and 832 feet in length include the angle $109° 17' 53''$.

Ans. 1353 feet.

14. In a triangle where the sides 3270 and 5456 yards in length include the angle $46° \ 18'$, the largest angle is $97° \ 13'$.

(2) Given an angle and two sides which do not contain it.

15. The sides being 3157·881, 3278·946, and the angle opposite the latter $63° \ 42' \ 25''$, the triangle is not ambiguous, the third side being 3052·915, and the other angles $59° \ 42' \ 17''$ and $56° \ 35' \ 18''$.

16. The sides being 426·893 feet, 249·873 feet, and the angle opposite the former being $105° \ 44' \ 49''$, the triangle is not ambiguous, the third side being 284·9 feet, and the other angles $39° \ 57' \ 42''$ and $34° \ 17' \ 19''$.

17. In a triangle *ABC*, the side *AB* is 345 feet, *AC* is 174 feet, and the angle *C* is $115° \ 36' \ 58''$. The angle *A* is $37° \ 20'$.

18. In a triangle *ABC* the angle *A* is $28° \ 17' \ 32''$, *AB* is 12 feet and *BC* is 15 feet. The remaining side *AC* is 24·45 feet.

19. The sides being 1249·688, 397·356 feet, and the angle opposite to the latter being $9° \ 19' \ 38''$, the triangle is ambiguous and the two solutions are—

(1) The remaining side 1575·2 feet, the angles $30° \ 38' \ 42''$, $140° \ 1' \ 40''$.

(2) The remaining side 891·3 feet, the angles $149° \ 21' \ 18''$, $21° \ 19' \ 4''$.

20. The sides being 579·8629, 482·7538 feet, and the angle opposite to the latter $48° \ 17' \ 23''$, the two solutions are—

(1) The remaining side 599·524, the angles $63° \ 43' \ 31''$, $67° \ 59' \ 6''$.

(2) The remaining side 172·116, the angles $116° \ 16' \ 29''$, $15° \ 26' \ 8''$.

IV. Three sides being given.

21. If the sides be 1654, 1664, 2706 feet, the largest angle is [illegible]° 17′.

[illegible]s of a triangle being 384, 279 and 609 yards, [illegible] is $132° \ 46' \ 44''$.

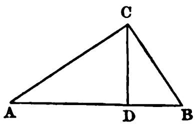
23. If the sides are 5284·763, 7399·841 and 6873·986 yards in length, the angles of the triangle are $73° 40' 48''$, $63° 4'$, $43° 15' 12''$.

24. If the sides of a triangle are 387, 516 and 650 yards, its largest angle is $90° 55' 44''$.

25. The sides of a triangle being 397, 426 and 368 feet in length. Find the angle which is neither the least nor the greatest. *Ans.* $59° 27' 58''$.

96. Area of an oblique-angled triangle.

To find the area of a triangle of which three parts are given, one at least being a side.



If AB be a side of the triangle ABC which is not less than either of the others, and CD be a perpendicular on AB, then with the notation hitherto used,

$$2 \text{ area of triangle} = AB \,.\, CD = bc \sin A,$$

$$\text{and since } \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \text{ (70)},$$

this expression has for its equivalents $ac \sin B$ or $ab \sin C$.

Hence, if an angle and the including sides are given, the area of the triangle can be computed. If an angle and other sides are given, or two angles and a side, the second of the including sides can be computed, and thus the area of the triangle can be found.

When three sides are given, since

$$c \,.\, CD = 2\sqrt{S(S-a)(S-b)(S-c)} \quad (93)$$

$$\therefore \text{ the area of the triangle } = \sqrt{S(S-a)(S-b)(S-c)}.$$

97. Ex. 1. Given two angles of a triangle $56° 27'$ and $73° 17' 29''$, and the side between them 1200 feet, to find the area of the triangle.

If A and B be the given angles of the triangle ABC, the remaining angle C is $50° 15' 31''$, and $AB = 1200$ feet.

$$2 \text{ area} = AB \,.\, AC \sin A \quad (96)$$

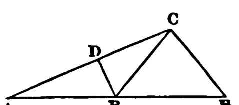
$$= AB^2 \,.\, \frac{\sin A \,.\, \sin B}{\sin C} \quad (70)$$

$$\text{area} = 720000 \, \frac{\sin A \sin B}{\sin C} \text{ square feet.}$$

$$\begin{array}{rl}
\log 720000 = & 5{\cdot}857\,3325 \\
L \sin A = & 9{\cdot}920\,8555 \\
L \sin B = & 9{\cdot}981\,2471 \\
 & \phantom{9{\cdot}981\,2}184 \\
L \sin C = & 9{\cdot}885\,8370 \\
 & \phantom{9{\cdot}885\,8}543 \\
 & 9{\cdot}885\,8913 \\
\text{ar. comp. } L \sin C = & {\cdot}114\,1087 \\
\log (\text{area}) = & 5{\cdot}873\,5622 \\
\log 747416 = & 5{\cdot}873\,5624
\end{array}$$

$\therefore$ area $= 747416$ square feet.

98. Ex. 2. If in a triangle ABC, $AC = 320$ feet, $BC = 196$ feet, and the angle CAB is $29°\ 13'\ 36''$, to find the area of the smaller of the two triangles which have those given parts.



Of the two triangles which these given parts present we are to confine our attention to the smaller one. In this from B draw BD perpendicular to AC. Then the area required is $\frac{1}{2} AC \,.\, BD = \frac{1}{2} AC \,.\, BC \sin ACB$.

Now $\sin B = \frac{320}{196} \sin A$,
whence $B = 52°\ 51'\ 37''$ or $127°\ 8'\ 23''$.

The condition of regarding the smaller triangle decides that $127°\ 8'\ 23''$ is the value of the angle ABC, and thence [illegible] $23°\ 38'\ 1''$.

$$\begin{aligned}\text{Then } \log(2 \text{ area}) &= 2\cdot505\,1500\\ &\ 2\cdot292\,2561\\ &\ \underline{\bar{1}\cdot603\,0214}\\ &\ 4\cdot400\,4275,\\ \text{whence } 2 \text{ area} &= 25143,\\ \text{and the area} &= 12572 \text{ square feet.}\end{aligned}$$

99. Ex. 3. If the three sides of a triangle are given, 8364, 6753, 7483 feet in length, to find the area.

$$\begin{aligned} 8364 &= a \quad \text{suppose,}\\ 6753 &= b\\ \underline{7483} &= c\\ 2S = 22600&,\\ S = 11300&.\end{aligned}$$

$$\begin{array}{rrrrrr} S = & 11300 & S = & 11300 & S = & 11300\\ a = & 8364 & b = & 6753 & c = & 7483\\ \hline S-a = & 2936 & S-b = & 4547 & S-c = & 3817\end{array}$$

$$\begin{aligned}\log S &= 4\cdot053\,0784\\ \log(S-a) &= 3\cdot467\,7561\\ \log(S-b) &= 3\cdot657\,7250\\ \log(S-c) &= \underline{3\cdot581\,7222}\\ &\ 14\cdot760\,2817\\ \therefore \log \text{area} &= 7\cdot380\,1409\end{aligned}$$

$$\therefore \text{area} = 23996110 \text{ square feet.}$$

100. *Examples for Practice.*

1. In a triangle where two sides 39 and 27 feet in length include the angle 112° 18′, the area is 487·18 square feet.

2. If the sides of an equilateral triangle are each 17 chains 4 links, the area is 12·57 acres.

3. The area of a triangle in which sides of the lengths 86 and 94 feet include the angle 127° 16′ 43″ is 3216 square feet.

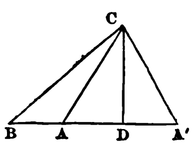
4. The area of a triangle wherein sides 84 and 73 feet in length include the angle $48° \ 27' \ 53''$ is 2295 square feet.

5. If the sides of a triangle are 826, 753 and 684 yards, its area is 24197 square yards.

6. The area of a quadrilateral whose diagonals 637 and 598 feet in length include the angle $37° \ 18'$ is 115418·6 square feet.

101. If a, b, B had been given to solve a triangle when b is less than a, and if c, c_1 be the values found for determining the third side, prove that $b^2 + c c_1 = a^2$.

Science Examination 1864.



Let CBA, CBA' be the two triangles with the given parts, BA and BA' being represented by c, c'. From C draw CD perpendicular to AA' and therefore bisecting it.

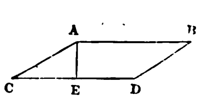
Then since AA' is bisected in D and produced to B

$$c c' + AD^2 = BD^2. \quad (Euc.\ ii.\ 6.)$$
$$\therefore c c' + AD^2 + CD^2 = BD^2 + CD^2,$$
$$\therefore c c' + b^2 = a^2.$$

102. In the ambiguous case where the given sides are 8 and 12 feet and the given angle 30°, find the difference of the areas of the two triangles which belong to these given elements. *Ans.* 31·748 square feet.

103. Find the area of a parallelogram whose sides (8 and 12 feet respectively) include an angle of 30°.

Science Examination 1867.



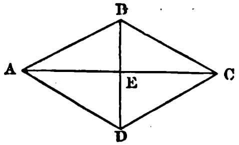
If $ABCD$ be the parallelogram wherein AC is 8 feet and CD is 12 feet, from A draw AE perpendicular to CD.

Then $AE = AC \sin 30° = 4$ feet,

and the area of the parallelogram $= AE \times CD$ (*Euclid*, i. 35)
$= 48$ square feet.

Generally if a, b be the lengths of the adjacent sides CD, AC of a parallelogram and C the angle ACD which they include, since the perpendicular $AE = b \sin C$, the area of the parallelogram is $ab \sin C$.

104. Each side of a rhombus is 50 feet, and one diagonal is double the other. Find the length in feet of either of the diagonals, and the angles of the figure.



Let AC, BD be the diagonals intersecting at right angles in E, AC being double BD. Then AB, BC, CD, DA are each 50 feet. Also AE the half of AC is double BE.

$$\text{Now, } 2500 = AB^2 = AE^2 + EB^2 = 5\ BE^2.$$

$$\therefore BE^2 = 500,\ BE = 22{\cdot}36 \text{ feet.}$$

$$\therefore AC = 89{\cdot}44 \text{ feet.}$$

$$\text{Again } \tan BAE = \frac{BE}{AE} = \tfrac{1}{2} = {\cdot}5,$$

$$\therefore BAE = 26^\circ\ 33'\ 54''.$$

$\therefore$ the angle BAD is $53^\circ\ 7'\ 48''$ and ABC is $126^\circ\ 52'\ 12''$.

105. Find the side of a square that shall be equal to a triangle whose sides are 7, 10, 12 feet respectively.

Science Examination 1867.

Let x be a side of the square in feet and x^2 accordingly the area of the square in square feet. Since then x^2 is the area of the triangle,

$$x^2 = \sqrt{14{\cdot}5 \times 2{\cdot}5 \times 4{\cdot}5 \times 7{\cdot}5}\ (96)$$

$$x = \sqrt[4]{14{\cdot}5 \times 2{\cdot}5 \times 4{\cdot}5 \times 7{\cdot}5} = 5{\cdot}979,$$

or 72 inches, to the nearest inch.

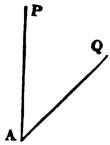
CHAPTER V.

HEIGHTS AND DISTANCES.

106. An important application of Trigonometry is to determine heights or distances which it is not convenient or possible to obtain by direct measurement. By measuring certain other lines and certain angles connected with these required heights or distances, triangles are formed, and by the solution of these triangles the required lengths are ascertained.

107. Some technical terms have first to be defined.

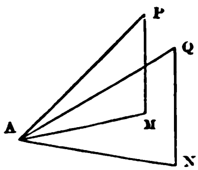
Def.—If P, Q be two points, and A some other point, and if AP, AQ be joined, the angle PAQ is called the angular bearing of P and Q from one another, or their angular distance, as they are seen at A. The same two points, therefore, have different angular bearings as they are seen from different points, and the angular distance is quite different from the linear distance, the line PQ.



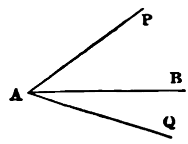
If AP is the direction towards the north of the meridian at the point A, the angle QAP is the bearing of Q from the meridian, east or west as the position of Q may be.

108. *Def.*—The plane made by the surface of standing water at any place is called the plane of the horizon, or the horizontal plane at that point, and a straight line drawn perpendicular to this horizontal plane is said to be vertical. An[illegible]through this vertical line is a vertical plane. There [illegible]y place innumerable vertical planes, but only [illegible] plane and one vertical line.

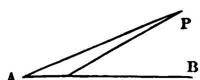
109. *Def.*—If A be a place of observation, P, Q two points, and if PM, QN be perpendiculars drawn on the horizontal plane at A, when MA, NA are joined the angle MAN is called the horizontal angular distance at A of P and Q, their angular distance being the angle PAQ.



110. *Def.*—If A be any place, P a point, and if the vertical plane at A which passes through P cuts the horizontal plane at A, the angle PAB is called the angular elevation of P above the horizon, as P is seen at A. Similarly, if Q be a point below the horizon and QAB be a vertical plane, the angle QAB is called the angular depression of Q below the horizon, as Q is seen at A.



In strictness, at every point on the earth's surface there will be a vertical line and a horizontal plane belonging to that particular point, but about the same place on the earth this variation from point to point is inappreciable, and vertical lines may be considered all parallel, and so may horizontal planes. Also when heavenly bodies are viewed, by reason of their great distance lines drawn to them from different points on the earth about the same place may be all considered parallel, inasmuch as, although they meet, they meet only at such a very great distance. Thus, if A, C be two points nearly at the same place on the earth, $PACB$ the vertical plane through P some heavenly body, the angular elevations of P above the horizon as seen at A and C are PAB and PCB respectively, and these differ by the angle APC, which is inappreciable by reason of the great length of PA or PC in comparison with AC.



111. The distance of points on the earth's surface when the ground between them is flat, be it horizontal or of uni-

form slope, can be measured by a Gunter's chain, or by a cord whose length is known by means of standard measures of yards or feet.

For measuring angles the principal instruments are the sextant or quadrant (different names for that which is substantially the same instrument) and the theodolite.

The powers of these instruments, however, are different.

The sextant or quadrant measures the angular distance between any two points, as these points are seen at the place of observation.

By the use, in addition, of an artificial reflecting horizon, the sextant can measure the angular elevation of a point above the observer's horizon.

The theodolite measures

(1) The horizontal angle between two points (109),

(2) The angular elevation or depression of any point from the horizon (110).

Thus if P and Q (figure of Art. 109) be two points, PM, QN perpendiculars from them on the horizontal plane at A,

the angle PAQ is measured by the sextant or quadrant only,

the angle MAN is measured by the theodolite,

the angles PAM, QAN are measured by the theodolite, or by the sextant with an artificial horizon.

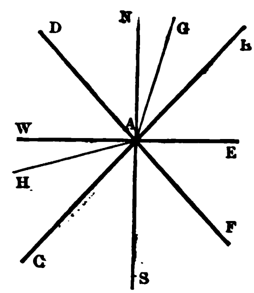
The construction and purpose of these instruments will be learned much better by an inspection of them than by the descriptions which are given in books.

The reader is now aware of the nature of the angles which it is in our power to measure readily, and in the following articles angles will be taken as measurable without stating in every case what is the instrument used.

112. *Def.*—The *bearing* of an object expresses the direction in which it is seen with respect to the points of the compass.

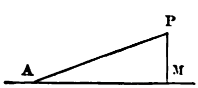
If A be any place of observation, AN the direction to the north, AS in the opposite direction is to the south, AE to the ri[illegible] when an observer looks northward, is the

east direction, and AW to the left hand is the west direction. NAS is the meridian at the point A which the sun's centre crosses at the apparent noon every day. Objects seen in the lines AN, AS, AE, AW, are said to be due north, south, east, west, of A respectively. If a line BAC is drawn bisecting the angle NAE, objects seen along the line AB are said to be to the north-east, and objects along AC to the south-west; while if DAF bisects the angle NAW, AD is the direction to the north-west and AF to the south-east. Intermediate directions are defined by their angular distances from the four principal directions. Thus if the angle NAG is 20°, objects seen in the line AG are said to bear 20° east of north. If WAH is 16°, objects seen in the line AH bear 16° south of west.



113. To find the height of a visible object above the horizontal plane when the foot of it can be approached.

Let PM be a vertical object, a wall or pillar for instance, whose height is required from the horizontal plane AM.



Suppose the base M to be accessible, measure a distance MA along the horizontal plane, and at A let the angle PAM be observed (111).

$$\text{Then } PM = MA \tan PAM,$$

whereby PM can be computed, logarithms if necessary being called into use.

Ex. Let AM be 100 yards, and the angle PAM be 12° 18′, the tangent of which is known from the tables to be ·2180353.

$$\therefore\ PM = 21{\cdot}80353 \text{ yards}$$
$$= 21 \text{ yards } 2 \text{ feet } 5 \text{ inches to the nearest inch.}$$

If the height PM were known, and the horizontal distance of a certain point A from it were required at which the angle PAM is measured,

$$MA = \frac{PM}{\tan PAM},$$

and the distance MA would then be known.

Obs.—The point A is in practice the observer's eye, and the height of the object from the ground on which he stands is to be ascertained by adding to PM the height of the observer's eye from the ground.

114. *Examples for Practice.*

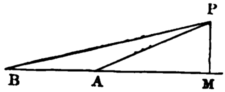
1. A and B are the tops of two poles, each 30 feet high. An observer at C in the same horizontal plane with the bases of these poles knows that CAB is a right angle, and measures the angular elevations (110) of A and B which are $34° 18'$ and $28° 14'$ respectively. Find AB the distance between the poles. *Ans.* $AB = 34\frac{1}{2}$ feet.

2. From the summit of a tower an object on the ground known to be 325 yards from its base has a depression (110) $8° 16'$. Find the height of the tower. *Ans.* 47·22 yards.

3. If the highest and lowest points of a vertical elevated line have the elevations $49° 27'$ and $38° 51'$ at a place at the horizontal distance 400 yards from it, find the length of the line. *Ans.* 145·4 yards.

[Find separately the heights above the horizontal plane of the ends of the line, and their difference is the length of the line.]

115. Suppose that M, the foot of the object PM whose height is required, cannot be approached. Let the angular elevation of P be first measured at a point A, and then at another point B, also on the horizon and in the same vertical plane $PAMB$, and let the length AB be measured. Then in the triangle PAB,



the angles PBA, PAB are known, and also the side AB, and from these AP can be computed (78).

Then $PM = PA \sin PAM$.

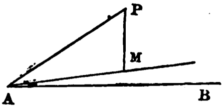
116. Ex. Let PAM be 30°, and PBM be 24° 17′, and the length AB 200 yards.

$$AP = 200 \frac{\sin 24^\circ\ 17'}{\sin\ 5^\circ\ 43'}.$$

$$PM = AP \sin 30^\circ = \tfrac{1}{2} AP.$$

$$\begin{aligned} \log 200 &= 2\cdot301\,0300 \\ L \sin 24^\circ\ 17' &= \underline{9\cdot614\,1051} \\ & \quad 11\cdot915\,1351 \\ L \sin 5^\circ\ 43' &= \underline{8\cdot998\,2994} \\ & \quad 2\cdot916\,8357 \\ \log 825\cdot72 &= 2\cdot916\,8328 \\ \therefore AP &= 825\cdot72 \text{ yards}, \\ PM &= 412\cdot86 \text{ yards}. \end{aligned}$$

117. To find the height of an accessible object which stands upon sloping ground.



Suppose the ground on which the vertical object PM stands has a uniform slope, its inclination to the horizon being a known angle which shall be called α.

Let MA be measured along the sloping ground in the vertical plane PAM, and the angular elevation of P, the angle PAB, be there observed. Then $PAM = PAB - \alpha$ is known; also $AMP = 90^\circ + \alpha$. Hence in the triangle APM we have two angles known, PAM and AMP, as well as the side AM between them, and PM can then be computed (78).

Ex. Let $\alpha = 5^\circ$, $MA = 80$ yards, $PAB = 25^\circ$. The angle $APM = 65^\circ$.

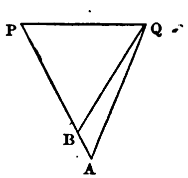
$$\frac{PM}{AM} = \frac{\sin 20^\circ}{\sin 65^\circ},$$

$$\log PM = \log AM + \text{L} \sin 20^\circ - \text{L} \sin 65^\circ$$

$$\begin{array}{rr}
= & 1{\cdot}9030900 \\
 & \underline{9{\cdot}5340517} \\
 & 11{\cdot}4371417 \\
 & \underline{9{\cdot}9572757} \\
 & 1{\cdot}4798660 \\
\log 30{\cdot}190 = & \underline{1{\cdot}4798631} \\
 & 29 \\
\text{prop. part for } 2 & 29
\end{array}$$

$\therefore$ $PM = 30{\cdot}1902$ yards.

118. To find the distance of two points, of which one is inaccessible.

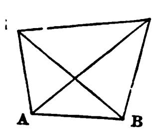


Let AP be a known line, and Q an inaccessible point, whose distance from P, an accessible point, is required.

At A let the angle PAQ be measured. Take a distance in AP to B, and measure the length AB. At B let the angle PBQ be measured.

Then in the triangle QAB, there are known the angles QBA and QAB, as well as the side AB between them. QB can therefore be computed (78). Then PB, the difference of the known lengths PA, BA being known, and QB, and also the angle PBQ, QP can be computed as a side of the triangle PBQ (81).

119. Let P, Q be two inaccessible points whose distance is required, and A, B two known points in the same plane with them, whose distance AB is known.



At A let the angles PAB, QAB be measured, and at B let the angles PBA, QBA be measured. Then

(1) In the triangle PAB there are known the angles PAB, PBA, and the side AB between them,

$\therefore$ AP can be computed (78).

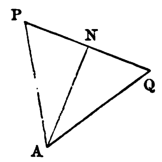
(2) In the triangle QAB there are known the angles QBA, QAB, and the side AB.

$$\therefore AQ \text{ can be computed.}$$

(3) In the triangle PAQ there are now known the sides AP, AQ, and the angle PAQ which they contain.

$$\therefore PQ \text{ can be computed (81).}$$

120. Let P, Q be two points whose distance is to be ascertained, while they are known to be at the same given distance from the point A. At A let the angle PAQ be measured.



Now a straight line AN bisecting the angle PAQ will also bisect PQ at right angles.

$$\therefore PQ = 2\,PN = 2\,PA \,.\, \sin\frac{PAQ}{2}.$$

Ex. $AP = AQ = 500$ yards,

$$PAQ = 58^\circ\ 29'.$$

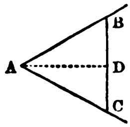
$$\therefore \frac{PAQ}{2} = 29^\circ\ 14', \text{ whose sine is } \cdot 4883674.$$

$$\therefore PQ = 488\cdot 3674 \text{ yards.}$$

121. Though some of the most obvious cases of determining heights and distances have thus been considered, yet problems of this class have such wide variety that the treatment of them can hardly be reduced to a few comprehensive cases, and several examples are therefore now offered with the working of them detailed.

122. Ex. 1. AB, AC are two straight railroads inclined at an angle $50^\circ\ 20'$. A locomotive engine starts from A along AB at the rate of 30 miles an hour. After an interval of an hour, another locomotive starts along AC at the rate of 45 miles an hour. Find the distance of the engines from each other three hours after the starting of the first.

If B and C be the position of the engines at the time prescribed, AB the length run by the first in three hours is 90 miles, and AC the length run by the second in two hours is also 90 miles, so that AB and AC are equal.



Join BC, and draw AD perpendicular to it. Then AD will bisect BC at right angles, and will also bisect the angle BAC.

$$\therefore\ BAD = 25^\circ\ 10',$$

$$BD = 90 \cdot \sin 25^\circ\ 10'$$

$$= 90 \times \cdot 4252528$$

$$= 38 \cdot 272752 \text{ miles.}$$

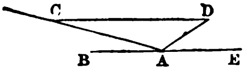
$\therefore$ BC the required distance is $76 \cdot 545504$,

or $76\frac{1}{2}$ miles nearly.

123. Ex. 2. At the foot of a hill a visible object has an elevation of $35^\circ\ 19'\ 18''$, and when the observer has walked 350 yards up the hill, away from the object, he finds himself on a level with it. The slope of the hill being 16° and the places of observation in the same vertical plane with the object, find the distance of the object from the first place of observation. *Science Examination* 1864.

In questions like this the computer is recommended to take some pains in drawing his figures with exactness. The geometrical relations of the parts will then more readily offer themselves to him, and his rough results, which measurement on his constructed figure gives, may check the more precise results of computation.

The construction in the present case will be this. After drawing BAE as the representative of a horizontal line, measure off the angle $BAC = 16^\circ$ the slope of the hill, and $AC = 350$ on some adopted scale, so that C is the observer's second position.



From the opposite side of the horizontal line measure off the angle $EAD = 35^\circ\ 19'\ 18''$, and then if CD is drawn parallel to BE, and therefore horizontal, D is the place of

the object. Then AD on the scale will give the distance required, with accuracy depending on the exactness of the drawing.

In computation we have in the triangle ACD the three parts,

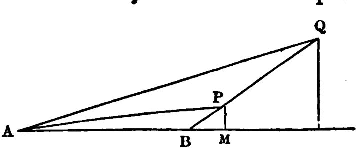
$$\text{angle } ACD = \text{angle } CAB = 16^\circ,$$
$$\text{angle } ADC = \text{angle } DAE = 35^\circ\ 19'\ 18'',$$
$$\text{and the side } AC = 350.$$

The case therefore is that considered in (80), and by the method of that article the side AD can be computed.

$$\frac{AD}{AC} = \frac{\sin 16^\circ}{\sin 35^\circ\ 19'\ 18''}.$$
$$\log AD = 2{\cdot}2223534$$
$$\therefore AD = 166{\cdot}86 \text{ feet.}$$

124. Ex. 3. The elevations of two mountains in the same line with the observer are $9^\circ\ 30'$ and $18^\circ\ 20'$. On approaching four miles nearer they both have an elevation of 37°. Find the heights of the mountains in yards.

To construct an accurate figure, along a line $ABMN$, supposed to be horizontal, take on any scale AB to represent 4 miles, or 7040 yards. At B make the angle $MBPQ = 37^\circ$. At A make the angles $BAP = 9^\circ\ 30'$ and $BAQ = 18^\circ\ 20'$. Then P and Q, the intersections of the lines thus drawn, represent the mountain tops, and if PM, QN be drawn perpendicular to the horizontal line these are the heights required.



To compute these heights,

(1) in the triangle ABP we have

$$\text{the angles } PAB = 9^\circ\ 30',$$
$$PBA = 143^\circ,$$
$$\text{the side } AB = 7040,$$

whence BP will be found to have for its logarithm $3{\cdot}4007763$.

$$\text{Then (2) } PM = BP \sin 37°,$$

$$\begin{aligned} \text{L} \sin 37° &= 9{\cdot}779|4630 \\ \log BP &= 3{\cdot}400|7763 \\ \therefore \log PM &= 3{\cdot}180|2393 \\ \log 1514{\cdot}4 &= 3{\cdot}180|2406 \\ \therefore PM &= 1514{\cdot}4 \text{ yards.} \end{aligned}$$

In the same manner, by solution of the triangle AQP, the logarithm of AQ, and thence QN, are found.

125. The following question may be similarly treated.

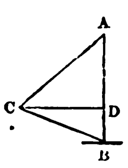
A person walking along a straight road observes a tall tree standing in front of a tower, both being in the road before him. The elevation of the top of the tower is 34° 15′ and of the top of the tree 25° 10′. On advancing 400 feet he finds the tower and the tree to have the same elevation 60° 15′. Find the height of the tower.

Ans. Height of the tower is 446 feet.

126. Ex. 4. From a window on one side of a street, which window was 30 feet above the horizontal ground, the height of a house on the opposite side of the street subtended an angle of 75°. The elevation of the top of this opposite house, as seen from the window, was 55°. Find the height of the house and the breadth of the street.

Let AB be the elevation of the house, C the place of observation. Draw CD perpendicular to AB. Then the observations are that the angle ACB is 75°, and the angle ACD is 55°, while DB, the height of C is 30 feet. Hence are to be determined AB the height of the house, and CD the breadth of the street.

Since $DCB = ACB - ACD = 20°$, $CBD = 70°$,

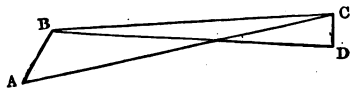


$$\begin{aligned} CD &= BD \,.\, \tan 70° \\ &= 30 \times 2{\cdot}7475 \\ &= 82{\cdot}42 \text{ feet,} \\ AD &= CD \tan 55° \\ &= 117{\cdot}7 \text{ feet.} \\ \therefore AB &= 147{\cdot}7 \text{ feet.} \end{aligned}$$

127. Ex. 5. To determine the height of the top C of a mountain a base AB of 2700 feet was measured in the horizontal plane, the angle subtended by CB at A was observed to be $50° 20'$, the angle subtended by AC at B was observed to be $110° 12'$, and the angle of elevation of C from B was observed to be $10° 7'$. Find the height of the mountain.

Let BD be the height of the mountain, the angle CBD being $10° 7'$.

Our object will be first to find CB from the given parts of the triangle CBA which the conditions of the problem present, where AB is 900 yards.



Then the angle BCA being $180° - 50° 20' - 110° 12'$ or $19° 28'$,

$$\frac{CB}{BA} = \frac{\sin 50° 20'}{\sin 19° 28'},$$

$$CB = 900 \frac{\sin 50° 20'}{\sin 19° 28'},$$

$$CD = CB \tan 10° 7',$$

$$\frac{CD}{900} = \frac{\sin 50° 20' . \tan 10° 7'}{\sin 19° 28'}.$$

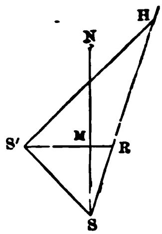
$$\therefore \log \frac{CD}{900} = \bar{1}\cdot 6150417,$$

$$\therefore \quad \frac{CD}{900} = \cdot 41213, \text{ and } CD = 370\cdot 92 \text{ yards.}$$

128. Ex. 6. From a ship at sea a rock and a headland are observed to be in the same straight line, which line has a bearing 18° east of north of the ship. After the ship has sailed six miles on its own course, which is north-west, the rock is observed from the ship to be due east, to the headland north-east. Find the distance of the rock from the headland.

Let S be the first position of the ship, SMN the line in

direction due north. Then if the angle NSR be constructed to be $18°$, SR is the direction in which rock and headland are seen from S.



Now take $NSS' = 45°$. Then SS' being north-west from S is the ship's direction, and if SS' be measured on some scale to represent six miles, S' is the ship's second position. If then (1) $S'MR$ be drawn due east, and therefore perpendicular to SN, and meets SRH in R, R is the position of the rock;

(2) if $MS'H$ be made $45°$, giving the north-east direction from S', and if $S'H$ meets SH in H, H is the headland, and RH is the distance to be found.

From this construction $SS'R$ and $RS'H$ being each $45°$, $SS'H$ is a right angle; and SS' and the angle $S'SH$ being known, SH can be computed.

Also $MS'H$ and MSS' being each $45°$, MS and MS' are equal and are known from SS'.

Then MSR being $18°$, SR can be computed, and RH the excess of SH over SR can be found.

This being the outline of the method to be adopted, we have

$$S'SH = 45° + 18° = 63°,$$

$$\therefore\ S'HS = 27°.$$

$$SH = \frac{SS'}{\sin S'HS} = \frac{6}{\sin 27°}$$

$$= \frac{6}{\cdot 45399} = 13\cdot 216 \text{ miles.}$$

$$\text{Again, } 36 = (SS')^2 = SM^2 + MS'^2 = 2SM^2,$$

$$SM = \sqrt{18},$$

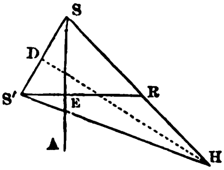
$$\text{and angle } SRM = 72°,$$

$$\therefore\ SR = \frac{SM}{\sin 72°} = \frac{\sqrt{18}}{\cdot 95106} = 4\cdot 46 \text{ miles.}$$

$$\therefore\ HR = SH - SR = 8\cdot 756 \text{ miles.}$$

129. Ex. 6. A ship at sea observes a rock and a headland both to bear in the direction south-east. The ship

sails for six miles in a direction 30° west of south, and it is then found that the rock bears due east of the ship and the headland 15° south of east. Find the distance of the rock from the headland.



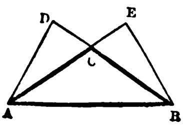
Let S be the first position of the ship, SA the south direction, SRH the line south-east, in which the rock and the headland are seen. Draw SS' making the angle $ASS' = 30°$, and if SS' represents six miles in length, S' is the ship's second place. From S' draw $S'ER$ cutting SA perpendicularly in E. This line is due east of S', and R, where it meets SH, is the rock. Draw $S'H$ making the angle $RS'H = 15°$ and H is the headland.

$$\text{Since } ESR = ERS = 45°,$$
$$SR = SE\sqrt{2}.$$
$$\text{Since } HSS' = 45° + 30° = 75°,$$
$$\text{and } HS'S = 60° + 15° = 75°,$$
$$\therefore \quad HS = HS',$$

and if HD is drawn perpendicular to SS' it will bisect SS', and make $SD =$ three miles.

$$\text{Now } SE = SS' \,.\, \sin 60° = 3\sqrt{3} \text{ miles},$$
$$\therefore \quad SR = SE\sqrt{2} = 7{\cdot}3485 \text{ miles}.$$
$$\text{Again, } DSH \text{ being } 75°,$$
$$DHS \text{ is } 15°.$$
$$\therefore \quad SH = \frac{SD}{\sin 15°} = \frac{3}{\sin 15°}$$
$$= 3 \times 3{\cdot}8637$$
$$= 11{\cdot}5911 \text{ miles}.$$
$$\therefore \quad RH = 4{\cdot}2426 \text{ miles}.$$

130. Ex. 7. A man in surveying a triangular field finds the base to be 100 yards long, and the perpendiculars from its extremities on the opposite sides respectively 58 and 60 yards long. Find the lengths of the sides.



If ABC be the triangle, AB its base of 100 yards, AD the perpendicular on $BC = 6$ yards, BE the perpendicular on $AC = 58$ yards.

Since $\sin A = \frac{BE}{BA} = \cdot 58$ and $\sin B = \frac{AD}{AB} = \cdot 6$,

the angles A and B can be found, and the angle C be consequently known. Then the sides of the triangle may be found.

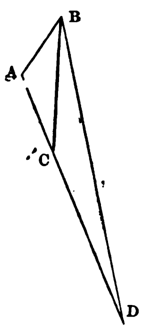
Since $\sin A = \cdot 58$, $A = 35° \, 27'$.
Since $\sin B = \cdot 6$, $B = 36° \, 52' \, 11''$.
$\therefore \quad C = 107° \, 40' \, 49''$.

$$\text{Now } AC = 100 \,.\, \frac{\sin B}{\sin C}.$$

$\therefore \quad AC = 62 \cdot 974$ yards.

In a similar manner BC may be found.

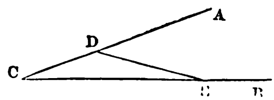
131. Ex. 8. A, B, C are three objects at known distances apart, namely, $AB = 1056$ yards, $AC = 924$ yards, $BC = 1716$ yards. An observer places himself at a station D from which C appears directly in front of A, and observes the angle $CDB = 14° \, 24'$. Find the distance CD.



First, in the triangle ACB where the sides are given the angle ACB can be computed (89).

Then in the triangle DCB, all the angles are known and also the side CB. Then DC can be computed (78) and will be found to be 2110 yards.

132. Ex. 9. C is a station and CA, CB are two directions at sea, including the angle $30° \, 20'$. In the line CA, 3 miles from C, a steamer is seen to start at the rate of 9 miles an hour. In what direction, relatively to CA, must the steamer start to be seen from C in the line CB after 40 minutes.



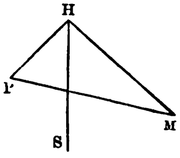
If D be the first and E the second position of the steamer, then CD is 3 miles, and from the given speed of the steamer and the time occupied in the passage, DE must be 6 miles.

The angle DCE being also known to be 30° 20′, the angle CDE is to be determined to give the direction of the steamer relatively to CA. This angle will be known if the angle DEC is found.

$$\text{Now } \sin DEC = \frac{CD}{DE} \sin DCE$$
$$= \tfrac{1}{2} \sin DCE = \cdot 2525149.$$
$$\therefore DEC = 14^\circ\ 37'\ 35''.$$
$$\therefore CDE = 180^\circ - (14^\circ\ 37'\ 35'' + 30^\circ\ 20')$$
$$= 135^\circ\ 2'\ 25''.$$

133. Ex. 10. During war a steam privateer was watching a harbour from a distance of $13\frac{1}{2}$ miles, and in a position south-west of the harbour. A merchantman sails from the harbour in a direction south-east at the rate of 9 miles an hour. In what direction and at what rate must the privateer steam to come up with the merchantman in two hours?

Let H be the harbour, HS the direction due south. If SHP and SHM be each 45°, they will be the south-west and south-east directions respectively. Then if HP represent $13\frac{1}{2}$ miles, P will be the original position of the privateer, and if HM represent 18 miles, the distance sailed by the merchantman in two hours, M will be the position of the merchantman when the privateer comes up with her, so that PM is the distance steamed by the privateer in two hours, and marks also the direction taken by the privateer.



If $4\frac{1}{2}$ miles be made the unit of length, $HM = 4$ and $HP = 3$. Hence PHM being a right angle,

$$PM = \sqrt{PH^2 + HM^2} = \sqrt{9+16} = 5,$$
or PM is $22\frac{1}{2}$ miles,

and the privateer must steam $11\frac{1}{4}$ miles an hour.

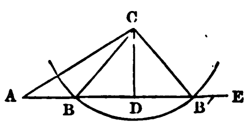
$$\text{Again } \sin HPM = \tfrac{4}{5} = \cdot 8.$$
$$\therefore \text{angle } HPM = 53^\circ\ 7'\ 48'',$$
$$= 45^\circ + 8^\circ\ 7'\ 48'',$$

or the direction PM is 8° 7′ 48″ south of east.

134. Ex. 11. A lighthouse is 12 miles distant from a ship and has a bearing north-east from the ship. The ship sails in a direction due east at the rate of 4 miles an hour. After what time will the distance of the ship from the lighthouse be 9 miles?

This problem exemplifies the ambiguous case (86) of oblique-angled triangles.

Let A be the first position of the ship, and ADE the line due east in which she is sailing. If the angle DAC be made 45°, AC is the direction north-east in which the lighthouse is seen, and if on some scale AC represents 12 miles, C will be the position of the lighthouse. If now on the scale of length used in measuring AC a distance 9 miles be taken, and a circle described from centre C with this radius, the circle will cut the line ADE in two points B, B' and these will mark the positions of the ship when it is at the distance of 9 miles from the lighthouse. We have now to find the distances AB and AB', and thence obtain the answer to the question by finding the times in which the ship sails from A to B and from A to B' at her assigned rate of 4 miles an hour.



From C draw CD perpendicular to AE, bisecting the line BB' at right angles.

Since CAD is 45°, so is ACD. $\therefore\ AD = DC$.

$$\therefore\ 144 = AC^2 = AD^2+DC^2 = 2AD^2.$$

$$\therefore\ AD = \sqrt{72} = DC.$$

$$\text{Again } 81 = CB^2 = CD^2+BD^2.$$

$$\therefore\ BD^2 = 81-72 = 9.$$

$$\therefore\ BD = 3 \text{ miles} = B'D.$$

$$\therefore\ AB = AD-BD = \sqrt{72}-3 = 5{\cdot}485 \text{ miles},$$

$$AB' = AD+B'D = \sqrt{72}+3 = 11{\cdot}485 \text{ miles}.$$

Then dividing these distances by 4 we have

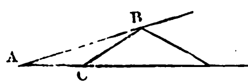
time of the ship from A to B = 1·371 hours,

" " " B' = 2·871 "

135. Ex. 12. The following problem offers another illustration of the ambiguous case.

Two railways intersect at the angle $35° 20'$. From the point of intersection two trains start together, one at the rate of 30 miles an hour. Find the rate of the other train so that after $2\frac{1}{2}$ hours the trains may be 50 miles apart.

If we can determine the distance travelled in $2\frac{1}{2}$ hours by the latter train, its speed is then known. Let A be the point of departure, $AB = 75$ miles the distance travelled by the first train. Let $BAC = 35° 20'$, so that ACD is the second railroad. If



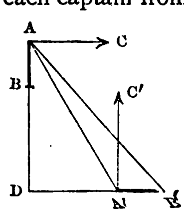
with centre B and radius 50 a circle be described it will cut ACD in two points C and D, and if the second train travels so as to be either at C or at D when the first train is at B, it will fulfil the condition of the distance of the two being then 50 miles.

By the process of solution used in (86),

$$AD = 86{\cdot}05 \text{ miles,}$$
$$AC = 36{\cdot}31 \quad ,,$$

If these spaces be described in $2\frac{1}{2}$ hours, the speed of the train is $34{\cdot}4$ or $14{\cdot}52$ miles an hour respectively.

136. Ex. 13. A captain marching on the left of his company observes another company of equal strength with his own, advancing from the right in a direction cutting his own line of march at right angles. He observes the angle subtended by this company at his eye to be $7° 30'$, and the angle between his own line of march and the line joining himself with the other captain, who is also at the left of his company, to be $60°$. Find the distance of each captain from the line of march of the other, the breadth of each company being 52 feet.



Let A be the observing captain, AB his company, and AC perpendicular to AB the direction in which he is marching,

A' the captain of the other company $A'B'$, and $A'C'$ the line in which he is

advancing. Produce AB and $B'A'$ to meet at right angles in D and join AA', AB'.

Then the facts before us are

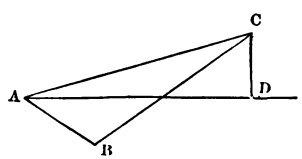
$$AB' = A'B = 52 \text{ feet},$$
$$\text{angle } CAA' = AA'D = 60°,$$
$$\text{angle } A'AB' = 7° \, 30',$$

and the distances to be computed are $A'D$ and AD.

These distances will result from the right-angled triangle $AA'D$ when AA' has been computed in the triangle $AA'B'$.

$$\text{The angle } AB'A' = 60° - 7° \, 30'$$
$$= 52° \, 30'.$$
$$\frac{AA'}{A'B'} \text{ or } \frac{AA'}{52} = \frac{\sin 52° \, 30'}{\sin 7° \, 30'}$$
$$\log 316{\cdot}06 = 2{\cdot}4997695$$
$$\therefore \quad AA' = 316 \text{ feet},$$
$$A'D = AA' \sin 30° = 158 \text{ feet},$$
$$AD = AA' \sin 60° = 273 \text{ feet}.$$

137. Ex. 14. An observer measures a base AB, 500 feet, in the same horizontal plane on which a tower CD stands, but not in the vertical plane passing through CD. At A he finds the elevation of C, the top of the tower, to be 30° 15′. He also observes the angle CAB, CBA to be respectively 60° 20′ and 56° 30′. Find the height of the tower.

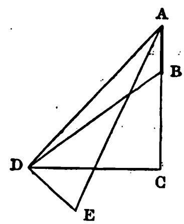


Since $CD = CA \sin 30° \, 15'$, the height of the tower will be known if CA be known. Now in the triangle CAB there are three given parts from which CA can be found.

$$\text{The angle } ACB \text{ being } 63° \, 10',$$
$$AC = AB \cdot \frac{\sin 56° \, 30'}{\sin 63° \, 10'},$$
$$CD = AB \cdot \frac{\sin 56° \, 30' \times \sin 30° \, 15'}{\sin 63° \, 10'}.$$

$$
\begin{aligned}
\text{Now } \log 500 &= 2{\cdot}6989700 \\
\text{L} \sin 56^\circ\ 30' &= 9.9211066 \\
\text{L} \sin 30^\circ\ 15' &= \underline{9{\cdot}7022357} \\
&\ 22{\cdot}3223123 \\
\text{L} \sin 63^\circ\ 10' &= \underline{9{\cdot}9505223} \\
\log CD &= 2{\cdot}3717900 \\
\log 23539 &= 2{\cdot}3717880 \\
\therefore\ CD &= 235{\cdot}39 \text{ feet.}
\end{aligned}
$$

138. Ex. 15. A is the top of a vertical column AB standing on a hill. BC is perpendicular to the horizontal plane on which a base DE, 380 feet, is measured. Find the height of AB from the following measured angles,



$ADC = 59^\circ$, $ADB = 26^\circ$,
$ADE = 70^\circ\ 20'$, $AED = 65^\circ\ 30'$.

In the triangle ADE we have adequate given parts for finding the side AD. Then AC and BC will be separately known.

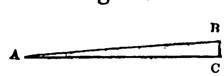
In the triangle ADE

$$\frac{AD}{DE} = \frac{\sin 65^\circ\ 30'}{\sin 44^\circ\ 10'}$$

$$
\begin{aligned}
\log AD &= 2{\cdot}6957308; \\
AC &= AD \tan 59^\circ \\
BC &= DC \tan 33^\circ, \\
&= AD \,.\, \sin 31^\circ \,.\, \tan 33^\circ. \\
\therefore\ AC &= 825{\cdot}95, \\
BC &= 165{\cdot}99,
\end{aligned}
$$

$AB = 660$ feet, if decimals be disregarded.

139. Ex. 16. A length being measured on a given uniform slope inclined at 3° to the horizon, to find the corresponding horizontal distance.



Let AB be a given distance measured along the slope,

AC a horizontal line, BC perpendicular to AC. The length of AC is required.

$$\text{Since } BAC = 3^\circ,\ ABC = 87^\circ.$$

$$\frac{AC}{AB} = \sin 87^\circ = \cdot 9986295,$$

$$AC = AB - \cdot 0013705 \, . AB.$$

Hence any length such as AB is to be reduced by $\cdot 0013705$ parts of itself to give the horizontal length AC.

If an incline is 1 in 100, what is the angle which the incline makes with the horizon?

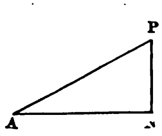
In the figure just used, if AB is 100 units of length, BC is 1 unit.

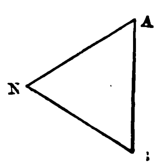
$$\therefore \sin BAC = \tfrac{1}{100} = \cdot 01,$$

$$\text{L} \sin BAC = 8.$$

This is an instance where the angle cannot be safely determined to seconds by a table which gives sines at intervals of minutes (39). To meet this difficulty tables are constructed of logarithmic sines at intervals of seconds for the first two or three degrees. When such a table is at hand we find that to the nearest second the angle BAC is $34' \; 23''$.

140. Ex. 17. A and B are two stations a mile apart, A due north of B. At the same instant a balloon is seen from A to bear $60^\circ \; 15'$ west of south, and as seen from B to bear $54^\circ \; 30'$ west of north; also the angle of elevation of the balloon as seen from A at the same time was $35^\circ \; 25' \; 25''$.





Find the perpendicular height of the balloon above the horizontal plane passing through A and B.

Let P be the position of the balloon. Draw PN perpen-

dicular to the horizontal plane meeting that plane in N. Join AN, BN, AB, AP.

Then in the triangle ANB the conditions of the question give

$$\text{the angle } NAB = 60^\circ\ 15',$$
$$\text{,,} \quad NBA = 54^\circ\ 30',$$
$$\text{and the side } AB = 1 \text{ mile},$$

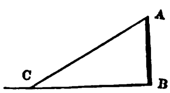
whence it may be computed that

$$AN = \cdot 89646 \text{ miles.}$$

Then in the triangle PAN, where the angle PAN is $35^\circ\ 25'\ 25''$,

$$PN = \cdot 637 \text{ miles.}$$

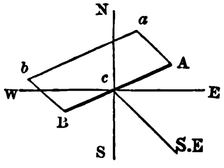
141. Ex. 18. If the length of the shadow of a vertical rod can be measured, and also the length of the rod, a right-angled triangle will give the direction in which the light is falling on the rod.



If AB be the vertical rod, BC its shadow, AC is the direction in which light is falling, and ACB is the angle which this direction makes with the horizon.

$$\text{Ex. } AB = 3 \text{ feet } 4 \text{ inches, } CB = 5 \text{ feet,}$$
$$\tan ACB = \frac{AB}{BC} = \tfrac{40}{60} = \tfrac{2}{3} = \cdot 6666,$$
$$\therefore ACB = 33^\circ\ 41'.$$

142. Ex. 19. If the altitude of the sun be known, and the aspect of a wall be also known, the size of the shadow of the wall at a given time of the day can be calculated.



Suppose that the sun be south-east, and AB be the base of a wall at a known inclination to the meridian. Let ab be the termination of its shadow. Let the altitude of the sun be 62°, and let the wall be 20 feet high.

Since a is the end of the shadow of one vertical edge of the wall

$$Aa = \frac{20}{\tan 62^\circ} = 10\cdot634 \text{ feet.}$$

Then the area of the shadow $AabB$ is the area of a parallelogram whose sides are

AB the length of the wall,
$Aa = 10\cdot634$ feet,

while the angle $aAB = ACS = 45^\circ + ACE$ is known from the given aspect of the wall.

Suppose that the position of this wall were required which would make this shadow have an area one-fourth of the area of the face of the wall.

Then $Aa \,.\, \sin aAB = 5$,

$$\sin aAB = \frac{5}{10\cdot634},$$

whence aAB is $61^\circ\ 57'$,
and ACE is $16^\circ\ 57'$.

143. Ex. 20. A tower 120 feet high stands in the middle of a field which is an equilateral triangle. From the top of the tower each side of the field subtends an angle of 100°. Find a side of the field.

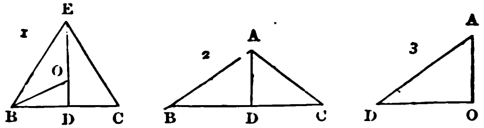


Figure 1 shows the field EBC in plan, and O the base of the tower. If EO be joined and produced to meet BC in D, BC is bisected in D, and if OB be joined the angle EBC is bisected by OB. If then x be the required length of a side of the field, $BD = \frac{x}{2}$; and each angle of the triangle being 60°, OBD is 30°.

Figure 2 is the plane through BC and A the highest point of the tower, wherein AB and AC are equal and the angle BAC is 100°.

Figure 3 is the elevation in a vertical plane through the points A, O, the top and bottom of the tower, and the point D.

$$\text{Now } OD = BD \tan 30° = \frac{x}{2\sqrt{3}},$$
$$AD^2 = AO^2 + OD^2 = AO^2 + \frac{x^2}{12}.$$

But the angle BAC being 100°, and AB, AC being equal to one another, ABD is 40°, and

$$AD = BD \tan 40° = \frac{x}{2} \tan 40°.$$
$$\therefore \frac{x^2}{4} \tan^2 40° = AO^2 + \frac{x^2}{12},$$
$$x^2 = \frac{4AO^2}{\tan^2 40 - \frac{1}{3}}$$
$$= \frac{4AO^2}{(\tan 40° + \frac{1}{3}\sqrt{3})(\tan 40° - \frac{1}{3}\sqrt{3})}.$$
$$\text{Now } \tan 40° = \cdot 8390996$$
$$\tfrac{1}{3}\sqrt{3} = \cdot 5773503.$$
$$\therefore x = \frac{240}{\sqrt{1 \cdot 4164499 \times \cdot 2617493}}.$$

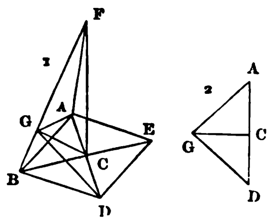
When the value of this fraction is computed by logarithms

$$x = 394 \cdot 15 \text{ feet}.$$

144. Ex. 21. In a pyramid on a square base each of the edges meeting in the vertex is double the length of a side of the base. Find the inclination to one another of two contiguous triangular faces. *Navigation Examination* 1864.

Let $ABDE$ be the square base, C the intersection of its diameters, F the vertex of the pyramid, FB an edge wherein the two triangular faces FAB, FAD intersect. Through AD let a plane pass perpendicular to FB and cutting it in

C. Then if AG, DG be joined, the angle AGD is the inclination required of two triangular faces.



The triangle AGD in its own plane is represented in Figure 2, and the angle AGD will be known if the angle AGC, the half of it, or the angle GAC can be determined.

Now if a be a side of the square base, the question states that $2a$ is an edge.

$$\text{Also } BC = \frac{a}{\sqrt{2}} = AC,$$

$$FC = \sqrt{4a^2 - \frac{a^2}{2}} = \sqrt{\frac{7}{2}}\,a$$

Since FGC is a right angle,

$$\frac{CG}{FC} = \sin BFC = \frac{BC}{BF} = \frac{1}{2\sqrt{2}}.$$

$$\therefore\ CG = \frac{\sqrt{7}}{4}a.$$

$$AG = \sqrt{CG^2 + AC^2} = \sqrt{\frac{7}{16} + \frac{1}{2}}\,a = \frac{\sqrt{15}}{4}a.$$

$$\therefore\ \sin GAC = \frac{CG}{AG} = \sqrt{\frac{7}{15}} = \sqrt{\cdot 466666}.$$

$$\text{Now } \log \cdot 46666\ldots = \bar{1}\cdot 6690069$$
$$\log \sqrt{\cdot 4666\ldots} = \bar{1}\cdot 8345035$$
$$\text{L} \sin GAC = 9\cdot 8345035,$$
$$\text{whence the angle } GAC = 43^\circ\ 5'\ 19''.$$
$$\therefore \text{ the angle } AGC = 46^\circ\ 54'\ 41'',$$
$$\text{and the angle } AGD = 93^\circ\ 49'\ 22''.$$

This is the angle required at which two trianglar faces of the pyramid are inclined to one another.

The inclination of one of the triangular faces to the base

is the complement of the angle whose sine is $\frac{1}{\sqrt{15}}$. This inclination is therefore determined and may be computed by the tables.

An edge such as BF makes with the base an angle whose sine is $\sqrt{\frac{7}{8}}$.

145. *Examples for Practice.*

1. A and B are two stations 250 yards apart. At A the observer finds an object C to be directly east of him, and measures the angle CAB, 34° 18′. At B the object C is directly south of him. Find the distance of A from C.

Ans. $206\frac{1}{2}$ yards.

2. A and B are two stations 437 yards apart. C is an object visible from each. The angles being measured $CAB = 24° \; 17'$, $CBA = 110° \; 34'$, find the distance CA.

Ans. $CA = 577$ feet.

3. ABC is a triangle wherein C is a right angle. Take P a point in AC and join BP. If it be known that AC is 600 yards, the angle BAC 20° 34′, and the angle BPC 54° 19′, find AP. *Ans.* $AP = 439$ yards.

4. A tower 30 yards high stands on elevated ground. An observer notes the elevations (110) of its base and summit to be 20° 13′ and 27° 14′ respectively. Find the height of the base of the tower above the observer's eye.

Ans. $75\frac{1}{2}$ yards.

5. At each of two stations A, B, 394 yards apart, an object C is observed, and the angles are measured, $CAB = 109° \; 17' \; 40''$, $CBA = 47° \; 39'$. Find the distance CA. *Ans.* $743\frac{1}{2}$ yards.

6. A and B are two objects 1180 yards asunder. C is a visible object in the same vertical plane with them and 50 feet below B. The angles are measured, $BAC = 30°$, $ABC = 60°$. Find the height of A above B.

Ans. 3004 feet.

7. A point C is 90 yards distant from a straight road in which are points A, B. The angles are measured, $CAB = 34° 17'$, $CBA = 46° 29'$. Find the distance between A and B. *Ans.* $217\frac{1}{2}$ yards.

8. An observer at a point A of a straight road AB sees two objects C, D in coincidence, and measures the angle $CAB = 34° 18'$. He advances 176 yards along the road to B until the angle CBA is observed to be $76° 10'$ and DBA is its supplement. Find the distance CD.

Ans. 72·87 yards.

9. AB is the front of a building and C is the end of a wall parallel to AB and 100 feet in front of it. There is a road parallel to AB and the wall, and at the point P in the road the end B of the building is just seen in a line with the end C of the wall, and at the point Q of the road the building just disappears behind the wall. If PQ is 126 feet, and the angle $CPQ = CQP = 53° 18'$, find the length of the building AB. *Ans.* 149 feet.

10. A and B are objects 260 yards apart. An observer at C knows his distance from the nearer object A to be 327 yards, and finds by measurement that the angle ACB is $35° 17' 28''$. Find his distance from B.

Ans. $445\frac{1}{2}$ yards.

11. AB is a vertical inaccessible object. In the horizontal plane through its base B an observer takes two positions C, D, where the summit A has the same elevation from the horizon $22° 18'$. He measures the length $CD = 328$ feet, and the angle $BCD = 35°$. Find the height AB.

Ans. 75·9 feet.

12. In a pyramid or tetrahedron, where each face is an equilateral triangle of the same size, find the angle which adjacent faces make with one another. *Ans.* $70° 32'$.

13. In a pyramid the faces which meet in the vertex are isosceles right-angled triangles, and the base is an equilateral triangle. Find the inclination to the base of one of the edges which meet in the vertex. *Ans.* $35° 16'$.

CHAPTER VI.

THE CIRCLE.

Area and Circumference of a Circle.

146. If r be the radius of a circle, the circumference of the circle is $2\pi r$, and the area of the circle is πr^2. The multiplier π is exhibited by an interminable decimal. Hence the area and circumference of a circle of given radius can never be found exactly, but may be found with as close an approach to exactness as we please by introducing more decimal places of π.

Euclid demonstrates that the areas of circles are proportional to the squares on their radii or diameters. Proof of the rest of the foregoing statement cannot be given on the elementary principles beyond which this book supposes that the reader has not yet advanced.

147. It may seem an unexpected confession that we are unable to estimate exactly the size and circumference of a figure so simple in form as a circle, and so frequently recurring in works of art, but the determination is not unattained because of its being beyond our limited mathematical powers, but because the object required is in itself unattainable. The problem of squaring the circle means the finding a square of the same size as a given circle. Now it can be demonstrated that the area of a circle is incommensurable with the square on its radius. Wherefore, if a certain unit of length be adopted, and to that unit the radius of the circle be expressed by any number, the area cannot be expressed exactly in the corresponding square unit by any integer, fraction, or terminated decimal. In like manner the radius and the circumfernce can be proved to be incom-

mensurable. In any unit of length to which one can be expressed, in this the other cannot be expressed in any finite form. If the diameter of a circle is 1 inch, the circumference is 3·14159 . . . inches, a decimal which will never end; while by continuing this decimal to more and more places the length of the circumference is approached more and more nearly.

148. Our inability to determine exactly the size or circumference of a circle is of no practical disadvantage, because these elements can be determined to any degree of nearness we desire, by continuing to more places the decimals which present themselves. It may be remembered, at the same time, that this want of perfect exactness is nothing more than arises whenever calculation passes beyond abstract given numbers to any data gained by measurement or observation. None of our measures or observations are exact. Until we can assign the time of any event with security of being correct beyond a certain decimal part of a second—until we can measure off a line and record its length with assurance of being correct beyond a certain decimal of an inch, we need not regret that we cannot compute to exactness the size or length of a circle from its given radius. The area or circumference of a circle can be found from its radius to more than a hundred places of decimals, whereas he must be a skilful observer with a fine instrument who can measure that radius, even to the millionth part of an inch.

149. The multiplier which gives the area of a circle from the square of its radius is the same as that which gives the circumference of a circle from its diameter. This multiplier is usually denoted by the symbol π.

It is generally sufficient to bear in memory the value of π to five places of decimals 3·14159, and to use as many of these places as the nature of the calculation may demand. The fraction $\frac{22}{7}$, which is equivalent to π as far as two places of decimals, will be often sufficiently exact.

150. The length of any arc of a given circle is known if the angle corresponding to it at the centre be known, for it must have the same ratio to the whole circumference as this angle has to 360°. (*Euclid*, vi. 33.)

151. The area of any sector of a given circle is known if either the angle corresponding to it at the centre be known, or the length of the arc belonging to it be known.

In the first case the sector has the same ratio to the area of the whole circle as the angle has to 360°.

In the second case, if a be the length of the arc, the sector has the same ratio to the area of the whole circle as this arc has to the circumference of the circle. (*Euclid*, vi. 33.)

$$\therefore \frac{\text{area of sector}}{\pi r^2} = \frac{a}{2\pi r}.$$

$$\text{area of sector} = \tfrac{1}{2}\, ar.$$

152. Under the same data the area of the segment of the circle is found by subtracting from the sector the area of the triangle formed by the bounding radii and the straight line which is the base of the segment.

153. *Examples.*

Ex. 1. If a circle has a diameter of 8 feet, the radius is 4 feet.

The circumference $= 8 \times 3{\cdot}14159 = 25{\cdot}13272$ feet.

The area $= 16 \times 3{\cdot}14159 = 50{\cdot}26544$ square feet.

Ex. 2. If the circumference of a circle is 20 feet,

$$\text{the radius} = \frac{10}{3{\cdot}14159} = 3{\cdot}183 \text{ feet.}$$

Ex. 3. A thin string is wound upon a cylinder of 1 foot radius. What length of string will be wound up in 50 revolutions.

Every revolution of the wheel winds on 2π feet of string: hence 50 revolutions wind on 100π or $314{\cdot}159$ feet

$$= 314 \text{ feet } 2 \text{ inches.}$$

Ex. 4. The driving-wheel of a locomotive engine has a diameter of 6 feet. How many revolutions does it make while the train runs a mile?

By the nature of rolling, in every revolution of the driving-wheel the train advances a distance equal to its circumference, or 2π yards. Hence in a mile the wheel makes $\frac{1760}{2\pi} = 280\cdot1$ revolutions, or turns round a little over 280 times.

Ex. 5. If the arc of a circle measures 50 feet while the radius is 18 feet, what is the arc of a sector whose base is the given arc. *Science Examination* 1867.

The area of the whole circle is $(18)^2\pi$ and its circumference 36π.

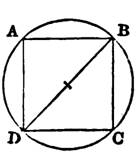
Now this sector must be a part of the whole area which is proportional to the arc, or is $\frac{50}{36\pi}$ of the area,

or $\frac{50 \times (18)^2}{36}$ square feet, or 450 square feet.

Ex. 6. If the circumference of a circle is 100 feet, what will be the area of the inscribed square?

Science Examination 1868.

Let $ABCD$ be the inscribed square, d the diameter BD of the circle.



Then area of square $= AB^2$

$= \frac{1}{2} BD^2 = \frac{1}{2}d^2$

But $100 = \pi d$, $d = \frac{100}{\pi}$.

$\therefore$ area of square $= \frac{10000}{2\pi^2}$

$= \frac{5000}{\pi^2}$.

$= 506\cdot606$ square feet,

$= 506$ square feet, 87 square inches.

Ex. 7. A wheel of 20 feet diameter turns uniformly on its

centre 100 times in a minute. At what speed is any point in its circumference moving?

In every revolution a point in the circumference travels through 20π feet; in a minute, therefore, it travels through 2000π or 6283·2 feet. This is a speed of rather more than 71 miles an hour.

Ex. 8. A curve in a railway has a radius of a mile. What is the length of the line between two points whose direct distance is 1100 yards?

If $2A$ be the angle at the centre which this circular arc subtends,

$$\sin A = \frac{550}{1760} = \cdot 3125,$$
$$\text{whence } A = 18° \ 12' \ 36'',$$
$$\text{and } \quad 2A = 36° \ 25' \ 12'',$$
$$\text{or } 131112 \text{ seconds.}$$

Now if the arc were completed into an entire circle, the whole circumference would be 3520π yards, and the circumference required is the portion of this corresponding to 13112 seconds when the whole circumference corresponds to 360°, or 1296000 seconds.

$$\text{Hence length of arc required} = \frac{131112 \times 352\pi}{129600} \text{ yards}$$
$$= 1118{\cdot}8 \text{ yards.}$$

Ex. 9. A field is in the form of the sector of a circle, its angle being 125° 15′ and radius 100 feet. Find the length of the arc of the sector and its area.

The circumference of the circle is 200π feet.

The arc required is the part of this which 125° 15′ is of 360°.

Its length then is $\frac{125\frac{1}{4}}{360} \cdot 200\pi = 178{\cdot}94$ feet.

The area of the circle is 10000π square feet.

$$\text{The area of the sector} = \frac{125\frac{1}{4}}{360} \cdot 10000\pi$$
$$= 10930 \text{ square feet.}$$

154. *Examples for Practice.*

1. If the radius of a circle is 71 inches, its circumference is 446·1 inches and its area is 15837 square inches.

2. The area of a circle being 155528 square feet, its radius is 222 feet 6 inches.

3. If the circumference of a circle is 16 yards, its area is 20·37 square yards.

4. If the angle of a sector of a circle be 60°, and the diameter of the circle be 200 inches, the area of the sector is 5236 square inches.

5. If the angle of a sector of a circle be 45°, and the diameter of the circle 250 feet, the area of the sector is 6136 square feet.

6. A sector of a circle is contained by two radii, each 6 feet long, and an arc of the circle 2 feet in length. The angle between the radii is 19° 5′ 53″.

7. In a circle whose radius is 1 yard, the area between the arc of a quadrant and the chord joining the ends of the arc is ·285 square yards.

155. *Annulus.*

An annulus is the space bounded by two circles which have the same point for their common centre. Its area is the difference between the areas of circles which have for their radii the outer and the inner radius of the annulus. If R be the outer radius, r the inner radius, the annulus is the area πR^2 with the area πr^2 cut out from it, or $\pi(R^2-r^2)$.

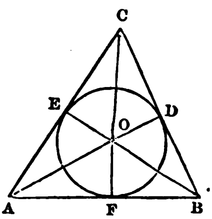
Ex. 1. If the outer and inner diameters of an annulus be 12 and 8 inches, the outer and inner radii are 6 and 4 inches, and the area of the annulus is $\pi(36-16)$, or 62·8 square inches.

Ex. 2. A circular plate of metal is 2 inches thick and 2 feet in diameter. What is the radius of the circular portion to be cut out so as to produce a ring weighing a pound, a c[illegible]f the metal weighing 4 ounces?

Let x be the inner radius of the annulus to be formed, its outer radius being 1 foot. The area of its flat surface is $\pi(1-x^2)$ square feet, and the ring has $2\pi(1-x^2)$ cubic inches. If it is to weigh a pound it must contain 4 cubic inches, or

$$2\pi(1-x^2) = 4,$$
$$1-x^2 = \frac{2}{\pi} = \cdot 6366,$$
$$x^2 = \cdot 3634,$$
$$x = \cdot 6 \text{ feet or } 7\cdot 2 \text{ inches.}$$

156. *Circle inscribed in or circumscribed about a triangle.*



To find the radius of the circle inscribed in a given triangle.

Let ABC be the given triangle, O the centre of the circle inscribed in it and touching it in the points D, E, F. Join AO, BO, CO; OE, OF, OD. Let r = radius of the circle.

Then $OD = OE = OF = r$.

$$\text{Thus triangle } COB = \tfrac{1}{2}OD \,.\, CB = \tfrac{1}{2}ra.$$
$$\text{,,} \quad AOC = \tfrac{1}{2}rb.$$
$$\text{,,} \quad BOA = \tfrac{1}{2}rc.$$

$\therefore$ the area of the triangle, being the sum of these three triangles, is $\frac{1}{2}(a+b+c)r$, or Sr if $2S$ be the perimeter of the triangle.

$$\therefore \sqrt{S(S-a)(S-b)(S-c)} = Sr \text{ (96)}$$

$$r = \frac{\sqrt{S(S-a)(S-b)(S-c)}}{S} = \sqrt{\frac{(S-a)(S-b)(S-c)}{S}}.$$

157. If the triangle be right-angled, C being the right angle, the area is $\frac{1}{2}ab$.

$$\therefore \tfrac{1}{2}(a+b+c)r = \tfrac{1}{2}ab,$$
$$r = \frac{ab}{a+b+c}.$$

158. If the triangle be equilateral or $a = b = c$, $S = \frac{3a}{2}$.

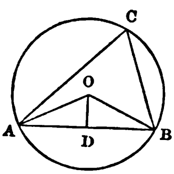
$$\therefore \tfrac{3}{2}ar = \text{area} = \sqrt{S(S-a)^3}$$

$$= \sqrt{\frac{3a}{2} \cdot \left(\frac{a}{2}\right)^3} = \frac{a^2}{4}\sqrt{3},$$

$$r = \frac{a}{2\sqrt{3}} = \frac{a\sqrt{3}}{6}.$$

159. To find the radius of the circle circumscribing a given triangle.

The centre of the circle circumscribing a triangle is constructed by bisecting any two sides of the triangle and drawing through the points of bisection perpendiculars to these sides. The intersection of these lines gives the centre of the circumscribing circle. (*Euclid*, iv. 5.)



If then O be the centre of the circle circumscribing the triangle ABC, and OD be drawn perpendicular to AB, $AD = DB$; also the angle $AOD = BOD$, and each of them being half the angle AOB is equal to the angle C. (*Euclid*, iii. 20.)

Hence if R be the radius of the circumscribing circle, a, b, c the sides of the triangle,

$$c = 2AD = 2AO \sin AOD = 2R \sin C,$$

$$\text{or } R = \frac{c}{2 \sin C} = \frac{b}{2 \sin B} = \frac{a}{2 \sin A} \quad (70).$$

160. If it be desirable to express the radius of the circumscribing circle in terms of the sides of the triangle, we have

$$R = \frac{abc}{4\sqrt{S(S-a)(S-b)(S-c)}} \quad (93).$$

161. When the triangle is right-angled, having C for its right angle, AB the hypothenuse is the diameter of the circumscribing circle. (*Euclid*, iv. 5, *Cor.*)

162. When the triangle is equilateral or $a=b=c$, $S=\dfrac{3a}{2}$

$$R=\frac{a}{\sqrt{3}}=\tfrac{1}{3}a\sqrt{3}.$$

163. Ex. 1. The sides of a triangular plate of metal are measured and found to be 10, 12, 14 inches in length, and the plate weighs 3 lbs. What is the weight of the largest circle that can be cut out from it?

The largest circle that can be cut out will be the inscribed circle. Let r be its radius in inches and πr^2 accordingly its area in square inches.

$$\text{Now } r=\frac{\text{area of triangle}}{S}.$$

$$\therefore \frac{\pi r^2}{\text{area of triangle}}=\pi\frac{\text{area of triangle}}{S^2}$$

$$=\pi\frac{\sqrt{(S-a)(S-b)(S-c)}}{\sqrt{S^3}},$$

and the weights of parts of the plate being as their areas,

$$\text{weight of circle cut out}=3\pi\sqrt{\frac{(S-a)(S-b)(S-c)}{S^3}}\text{ lbs.}$$

$$\text{Now } S=18,\ S-a=4,\ S-b=6,\ S-c=8.$$

$$\therefore \text{ weight of circle cut out}=3\pi\sqrt{\frac{4\times6\times8}{18^3}}=\frac{\pi}{3}\sqrt{\frac{8}{3}}$$

$$=\frac{\pi}{9}\sqrt{24}=1\cdot69\text{ lbs.}$$

Ex. 2. If out of a triangle whose sides are 10 feet the inscribed circle be taken away, what is the area of the remaining portion? *Science Examination* 1868.

The radius of the inscribed circle is $\dfrac{10\sqrt{3}}{6}$ or $\dfrac{5\sqrt{3}}{3}$ square feet, and the area of the triangle is $25\sqrt{3}$ square feet or 43·30 square feet.

The area of the inscribed circle is $\frac{25\pi}{3}$ or 26·179 square feet.

∴ the area of the remaining portion is 17·122 square feet.

Ex. 3. What is the diameter of the smallest circular plate from which it is possible to cut an equilateral triangle whose sides are each 9 inches long?

The circular plate must circumscribe the required triangle. Its radius is accordingly $3\sqrt{3}$, or 5·196 inches, and its diameter is 10·39 inches.

164. *Examples for Practice.*

1. The radius of a circle inscribed in a triangle whose sides are 329, 340 and 331 feet is 96·165 feet.

2. Find the diameter of a circle circumscribing a triangle wherein a side 100 feet long has the angle 27° 34′ opposite to it. *Ans.* 216 feet.

3. If the perpendicular sides of a right-angled triangle are 12 and 17 feet in length, the area of the circle circumscribing the triangle is 340 square feet.

4. What is the radius of a circle where the inscribed equilateral triangle has an area of a square yard? *Ans.* 1·32 feet.

5. In a right-angled isosceles triangle the radius of the inscribed circle is 1 foot. The length of each of the shorter sides is 3·414 feet.

6. Find the circumference of a circle inscribed in a right-angled triangle wherein the perpendicular sides are 21 and 28 feet in length. *Ans.* 42 feet nearly.

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